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Cross-entropy measure of uncertain variables

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ABSTRACT

ross-entropy is a measure of the difference between two distribution functions. In order to deal with the divergence of uncertain variables via uncertainty distributions, this paper aims at introducing the concept of cross-entropy for uncertain variables based on uncertain theory, as well as investigating some mathematical properties of this concept. Several practical examples are also provided to calculate uncertain cross-entropy. Furthermore, the minimum cross-entropy principle is proposed in this paper. Finally, a study of generalized cross-entropy for uncertain variables is carried out.

1. Introduction

Probability theory, fuzzy set theory, rough set theory, and credibility theory were all introduced to describe non-deterministic phenomena. However, some of the non-deterministic phenomena expressed in the natural language, e.g. "about 100 km", "approximately 39 °C", "big size", are neither random nor fuzzy. Liu [16] founded uncertainty theory, as a branch of mathematics based on normality, self-duality, countable subadditivity, and product measure axioms. An uncertain measure is used to indicate the degree of belief that an uncertain event may occur. An uncertain variable is a measurable function from an uncertainty space to the set of real numbers and this concept is used to represent uncertain quantities. The uncertainty distribution is a description of an uncertain variable. Uncertainty theory has wide applications in programming, logic, risk management, and reliability theory. In many cases, the uncertainty is not static, but changes over time. In order to describe dynamic uncertain systems, uncertain processes were first introduced by Liu [17]. Uncertain statistics is a methodology for collecting and interpreting experimental data (provided by experts) in the framework of uncertainty theory.

Suppose that we know the states of a system take values in a specific set with unknown distribution, although we do not know the exact form of this distribution function. However, we may learn constraints on this distribution: expectations, variance, or bounds on these values. Suppose that we need to choose a distribution that is in some sense the best estimate given what we know. Usually there are infinite many distributions satisfying the constraints. Which one should we choose? Before answering this question, we will first discuss the concepts of entropy and cross-entropy.

In 1949, Shannon [25] introduced entropy to measure the degree of uncertainty of random variables. Inspired by the Shannon entropy, fuzzy entropy was proposed by Zadeh [34] to quantify the amount of fuzziness, and the entropy of a fuzzy event is defined as a weighted Shannon entropy. Fuzzy entropy has been studied by many researchers such as [7,12,14,15,20–22,33,36]. The principle of maximum entropy was proposed by Jaynes [11]: of all the distributions that satisfy the constraints, choose the one with the largest entropy. Besides this method, cross-entropy was introduced by Good [8].

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It is a non-symmetric measure of the difference between two probability distributions. Other names of this concept include expected weight of evidence, directed divergence, and relative entropy. Based on De Luca and Termini's [7] fuzzy entropy, Bhandari and Pal [1] defined the cross-entropy for a fuzzy set via its membership function. The theory of fuzzy cross-entropy has been studied in [2,24]. The principle of minimum cross-entropy was proposed by Kullback [13]: from the distributions that satisfy the constraints, choose the one with the least cross-entropy.The principle of maximum entropy can be used to select a number of representative samples from a large database [28]. The principle of maximum entropy and principle of minimum cross-entropy have been applied to machine learning and to decision trees; see [10,26,27,29–31,35] for details. Other applications include portfolio selection [23] and optimization models [9,32].

Uncertainty theory is used to model human uncertainty. The uncertainty distribution plays a fundamental role. Unlike probability distribution (based on the sample), we often ask some domain experts to evaluate their degree of belief that each event will occur. Then the empirical prior information becomes more important. In many real problems, the distribution function is unavailable except for partial information, for example, prior distribution function, which may be based on intuition or experience with the problem. In order to better estimating the uncertainty distribution, Liu [19] introduced uncertain entropy to characterize uncertainty resulting from information deficiency. Chen and Dai [4] investigated the maximum entropy principle of uncertainty distribution for uncertain variables. To compute the entropy more conveniently, Dai and Chen [5] provide some formulas for the entropy of functions dealing with uncertain variables with regular uncertain distributions. In order to deal with the divergence of two given uncertain distributions, this paper will introduce the concept of cross-entropy for uncertain variables. Several practical examples are also provided to calculate uncertain cross-entropy. In practice, we often need to estimate the uncertainty distribution of an uncertain variable from the known (partial) information, for example, prior uncertainty distribution, which may be based on intuition or experience with the particular problem. In this context, the *principle of minimum cross-entropy* in uncertainty theory will be studied. The rest of the paper is organized as follows: some preliminary concepts of uncertainty theory are briefly recalled in Section 2. The concept and basic properties of entropy of uncertain variables are introduced in Section 3. The concept of cross-entropy for uncertain variables is introduced in Section 4, where some mathematical properties are also studied. The minimum cross-entropy principle theorem for uncertain variables is proved in Section 5. A study of generalized cross-entropy for uncertain variables is carried out in Section 6. Finally, a brief summary is given in Section 7.

2. Preliminaries

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. An uncertain measure $\mathcal{M}$ [16] is a set function defined on $\mathcal{L}$ satisfying the following four axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leqslant \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$$

Axiom 4. (Product Measure Axiom) Let Γ_k be nonempty sets on which $\mathcal{M}_k$ are uncertain measures, $k = 1,2,\ldots,n$, respectively. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{n}\Lambda_k\right\} = \min_{1\leqslant i\leqslant n}\mathcal{M}_k\{\Lambda_k\}$$

An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. The uncertainty distribution function $\Phi : \Re \to [0,1]$ of an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \leqslant x\}$. The expected value operator of uncertain variable was defined by Liu [16] as

$$E[\xi] = \int_0^{+\infty}\mathcal{M}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0}\mathcal{M}\{\xi \leqslant r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined as $E[(\xi - e)^2]$, where e is the finite expected value of ξ.

3. Entropy of uncertain variables

Definition 1 (*Liu [19]*). Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x$$