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The Quintuple Implication Principle of fuzzy reasoning

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ABSTRACT

Fuzzy Modus Ponens (FMP) and Fuzzy Modus Tollens (FMT) are two fundamental patterns of approximate reasoning. Suppose A and B are fuzzy predicates and "IF A THEN B" is a fuzzy rule. Approximate reasoning often requires to derive an approximation B^* of B from a given approximation A^* of A, or vice versa. To solve these problems, Zadeh introduces the well-known Compositional Rule of Inference (CRI), which models fuzzy rule by implication and computes B^* (A^*, resp.) by composing A^* (B^*, resp.) with $A \to B$. Wang argues that the use of the compositional operation is logically not sufficiently justified and proposes the Triple Implication Principle (TIP) instead. Both CRI and TIP do not *explicitly* use the closeness of A and A^* (or that of B and B^*) in the process of calculating the consequence, which makes the thus computed approximation sometimes useless or misleading.

In this paper, we propose the Quintuple Implication Principle (QIP) for fuzzy reasoning, which characterizes the approximation B^* of B (A^* of A, resp.) as the formula which is best supported by $A \to B$, $A^* \to A$ and A^* ($A \to B$, $B \to B^*$ and B^*, resp.). Based upon Monoidal t-norm Logic (MTL), this paper applies QIP to solve FMP and FMT for four important implications. Most importantly, we show that QIP, when using Gödel implication, computes exactly the same approximation as Mamdani-type fuzzy inference does. This is surprising as Mamdani interprets fuzzy rules in terms of the minimum operation, while CRI, TIP and QIP all interpret fuzzy rules in terms of implication.

1. Introduction

Fuzzy Modus Ponens (FMP) and Fuzzy Modus Tollens (FMT) (see Table 1) are two fundamental patterns of approximate reasoning. Given a fuzzy rule "IF A THEN B", approximate reasoning often requires to derive an approximation B^* of B from a given approximation A^* of A, or vice versa. For example, suppose A represents that "*x is tall*" and B represents that "*x is heavy*", and "IF A THEN B" is a fuzzy rule with A, B interpreted as before. Let A^* represent "*x is over* 175 cm." Then what can we infer from A^* and "IF A THEN B"? How good an approximation of B can we get? Such a reasoning problem can be formulated as a FMP problem. Similarly, if we have an approximation B^* of B, e.g. "*x weighs over* 75 kg," how good an estimation can we get from B^* and "IF A THEN B"? Such a reasoning problem can be formulated as a FMT problem.

To solve these reasoning problems, Zadeh proposed the well-known Compositional Rule of Inference (CRI) [20] in 1973. The CRI models fuzzy predicates as fuzzy sets and models fuzzy rules as fuzzy relations. Suppose A is a fuzzy predicate over U, and B a fuzzy predicate over V. The fuzzy rule "IF A THEN B" is represented as a fuzzy relation

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Table 1
Fuzzy Modus Ponens (left) and Fuzzy Modus Tollens (right).

Rule	IF A THEN B	Rule	IF A THEN B
Premise	A^*	Premise	B^*
Compute	B^*	Compute	A^*
	(**FMP**)		(**FMT**)

$$R(x,y) = (1 - A(x)) \vee (A(x) \wedge B(y)). \tag{1}$$

The CRI solution of FMP is obtained by combining A^* with R by using the sup-min composition:

$$B^*(y) = (A^* \circ R)(y) = \bigvee_{x \in U} (A^*(x) \wedge R(x,y)) \tag{2}$$

The CRI solution of FMT is obtained in a similar way:

$$A^*(x) = (R \circ B^*)(x) = \bigvee_{y \in V} (B^*(y) \wedge R(x,y)) \tag{3}$$

The fuzzy relation R and the composition operation $\circ$ in (2) and (3) can be replaced by other relations/operations. Let $\rightarrow_Z$ be the binary operation on [0, 1] defined as

$$a \rightarrow_Z b = (1 - a) \vee (a \wedge b) \tag{4}$$

It is easy to verify that $\rightarrow_Z$ is an implication operation (cf. Section 2), and the fuzzy relation R in (1) can be represented as

$$R(x,y) = A(x) \rightarrow_Z B(y). \tag{5}$$

That is, the fuzzy rule "IF A THEN B" is represented as an implication $A(x) \rightarrow_Z B(y)$. In general, we could define the fuzzy relation R in terms of any implication operator $\rightarrow$ instead of $\rightarrow_Z$ in (5) and combine the minor premise A^* (or B^*) with the major premise $A \rightarrow B$ by using any triangle norm (t-norm for short) $\otimes$ instead of $\wedge$ in (2) and (3). Therefore, the general CRI solutions of FMP and FMT are

$$B^*(y) = \bigvee_{x \in U} (A^*(x) \otimes R(x,y)) = \bigvee_{x \in U} (A^*(x) \otimes (A(x) \rightarrow B(y))), \tag{6}$$

$$A^*(x) = \bigvee_{y \in V} (B^*(y) \otimes R(x,y)) = \bigvee_{y \in V} (B^*(y) \otimes (A(x) \rightarrow B(y))), \tag{7}$$

where $\otimes$ is a t-norm, $\rightarrow$ is an implication operator.

Unlike the general CRI model, Mamdani [10] interprets a fuzzy rule in terms of the minimum operation (sometimes called Mamdani implication) and uses the same sup-min composition. Let

$$R_M(x,y) = A(x) \wedge B(y). \tag{8}$$

The Mamdani-type solutions of FMP and FMT are defined in (9) and (10) respectively.

$$B^*(y) = \bigvee_{x \in U} (A^*(x) \wedge R_M(x,y)) = \bigvee_{x \in U} (A^*(x) \wedge (A(x) \wedge B(y))), \tag{9}$$

$$A^*(x) = \bigvee_{y \in V} (B^*(y) \wedge R_M(x,y)) = \bigvee_{y \in V} (B^*(y) \wedge (A(x) \wedge B(y))). \tag{10}$$

While the Mamdani-type fuzzy reasoning has been very successful in applications of fuzzy control systems, the use of the minimum operation for interpreting fuzzy rules is, at first glance, contrary to intuition, as minimum is not extension of the implication operator in the classical Boolean logic. Although several authors have discussed the rationality of the choice of the minimum operation (cf. e.g. [1,9]), it remains unclear how to interpret Mamdani-type fuzzy reasoning from the viewpoint of fuzzy logic.

The compositional operation used in CRI (see (2)) is regarded by Wang [18] as logically not sufficiently justified. He proposes the Triple Implication Principle (TIP), which characterizes the consequence as the one which is best supported by the rule $A \rightarrow B$ and the premise. To formally express the intuitive notion of "the best supported consequence", Wang [18] develops a quasi-propositional deductive system $\mathcal{L}^*$ and defines the best supported consequence in terms of tautology. The TIP solutions of FMP and FMT are very similar to the corresponding CRI solutions, but the TIP also provides a criteria for choosing the appropriate t-norm and implication operator.

When making approximate reasoning, e.g. solving FMP, it is natural to believe that B^* should be *close* to B if A^* is *close* to A. For example, let x be a variable and $y = f(x)$ a continuous function of x. Suppose we know that $f(1) = 1$. Then it is reasonable to guess that $f(1.01)$ is very close to 1, but it is meaningless to estimate the value of $f(10^{10})$ from $f(1) = 1$. Similarly, when computing a solution of FMP, the relation between A and A^* should be also taken into account. In both CRI and TIP, however, the closeness between A and A^* is not *explicitly* used in the process of calculating the consequence. Shortly we will see in Example 1 that this somehow makes the thus computed solution useless or misleading in some cases.