Information Sciences 297 (2015) 236–253

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Pairwise comparison matrix with fuzzy elements on alo-group

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A R T I C L E I N F O

Article history:
Received 7 February 2014
Received in revised form 7 October 2014
Accepted 7 November 2014
Available online 15 November 2014

Keywords:
Multi-criteria optimization
Pair-wise comparison matrix
Fuzzy element
Alo-group

A B S T R A C T

This paper deals with pairwise comparison matrices with fuzzy elements. Fuzzy elements of the pairwise comparison matrix are applied whenever the decision maker is not sure about the value of his/her evaluation of the relative importance of elements in question. In comparison with pairwise comparison matrices with crisp elements investigated in the literature, here we investigate pairwise comparison matrices with elements from abelian linearly ordered group (alo-group) over a real interval. We generalize the concept of reciprocity and consistency of crisp pairwise comparison matrices to matrices with triangular fuzzy numbers (PCFN matrices). We also define the concept of priority vector which is a generalization of the crisp concept. Such an approach allows for a generalization dealing both with the PCFN matrices on the additive, multiplicative and also fuzzy alo-groups. It unifies several approaches known from the literature. Moreover, we also deal with the problem of measuring the inconsistency of PCFN matrices by defining corresponding indexes. The first index called the consistency grade G is the maximal alpha of alpha-cut, such that the corresponding PCFN matrix is still alpha-consistent. On the other hand, the consistency index I of the PCFN matrix measures the distance of the PCFN matrix to the closest ratio matrix. If the PCFN matrix is crisp and consistent, then G is equal to 1 and the consistency index I is equal to the identity element e of the alo-group, otherwise, G is less than 1, or I is greater than e. Four numerical examples are presented to illustrate the concepts and derived properties. Finally, we show that the properties of reciprocity and consistency of PCFN matrices are saved by special transformations based on fuzzy extensions of isomorphisms between two alo-groups.

1. Introduction

Fuzzy elements of the pairwise comparison matrix (PCF matrix) can be applied whenever the decision maker (DM) is not sure about the preference degree of his/her evaluation of elements in question. A *decision making problem (DM problem)* can be formulated as follows:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives ($n > 2$). The aim is to rank the alternatives from the best to the worst (or, vice versa), using the information given by the DM in the form of an $n \times n$ PCF matrix.

The DM can acknowledge fuzzy pairwise preference data as imprecise knowledge about regular preference information. The fuzzy interval preference matrix is then seen as constraining an ill-known precise consistent comparison matrix. Inconsistencies in comparison data are thus explicitly explained by the imprecise nature of human-originated information.

An ordinal *ranking* of alternatives is required to obtain the best alternative(s), however, it often occurs that the DM is not satisfied with the ordinal ranking among alternatives and a cardinal ranking i.e. rating is also required.

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http://dx.doi.org/10.1016/j.ins.2014.11.010
0020-0255/

In the recent literature we can find papers dealing with applications of pairwise comparison method where evaluations require fuzzy quantities, for instance when evaluating regional projects, web pages, e-commerce proposals, etc., see e.g. [6,27,2,18,15,14]. In the papers [5,7], the authors proposed a method for measuring inconsistency of fuzzy pair-wise comparison matrix based on Saaty's principal eigenvector method. However, this method is rather cumbersome and numerically difficult. The earliest work in AHP using fuzzy quantities as data was published by van Laarhoven and Pedrycz [26]. They compared fuzzy ratios described by triangular membership functions. The method of logarithmic least squares was used to derive local fuzzy priorities. Later on, using a geometric mean, Buckley et al. [4] determined fuzzy priorities of comparison ratios whose membership functions were assumed trapezoidal. The issue of consistency in AHP using fuzzy sets as elements of the matrix was first tackled by Salo in [25]. Departing from the fuzzy arithmetic approach, fuzzy weights using an auxiliary mathematical programming formulation describing relative fuzzy ratios as constraints on the membership values of local priorities were derived. Leung and Cao, see [16], proposed a notion of tolerance deviation of fuzzy relative importance that is strongly related to Saaty's consistency ratio, see [24,12,13].

The former works that solved the problem of finding a rank of the given alternatives based on some PCF matrix are [16–20,28]. In [28] some simple linear programming models for deriving the priority vectors from various interval fuzzy preference relations are proposed. Leung and Cao [16] proposed a new definition of the PCF reciprocal matrix by setting deviation tolerances based on an idea of allowing inconsistent information. Mahmoudzadeh and Bafandeh [17] further discussed Leung and Cao's work and proposed a new method of fuzzy consistency test by direct fuzzification of QR (Quick Response) algorithm which is one of the numerical methods for calculating eigenvalues of an arbitrary matrix. Ramik and Korviny in [23] investigated inconsistency of pairwise comparison matrix with fuzzy elements based on geometric mean. They proposed an inconsistency index which, however, does not measure inconsistency as well as uncertainty ideally.

The structure of the paper is as follows. In Sections 2 and 3 we present some basic concepts and ideas of PCF matrices with elements being (L,R)-fuzzy numbers of the alo-group over a real interval (PCFN matrices). In Section 4 some methods for deriving priority vectors from such matrices are proposed. The results presented in Sections 3 and 4 are new and have not been published yet. Moreover, Section 4 consists of four subsections dealing with the particular character of the alo-groups of the real line: additive, multiplicative, fuzzy additive and fuzzy multiplicative one. In Section 5, eight illustrative numerical examples are presented and discussed. In Section 6, we show that the new properties of reciprocity and consistency of PCFN matrices are saved by special transformations based on fuzzy extensions of isomorphisms between two alo-groups. This result can be applied to all isomorphisms between 4 important alo-groups on the real line. Summary of the results closes the paper.

2. Abelian linearly ordered groups

In this paper we shall investigate pairwise comparison matrices with elements being fuzzy quantities of the real line **R**. Particularly, we shall deal with PCF matrices where the elements are triangular fuzzy numbers of some commutative linearly ordered group over a real interval. Moreover, we assume that all diagonal elements of the matrix are crisp, particularly they are equal to the identity element the group, see bellow:

$$\widetilde{A} = \begin{bmatrix} e & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & e & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & e \end{bmatrix}. \tag{1}$$

In this section, we recall some notions and properties related to abelian linearly ordered groups. Such an approach allows for unifying the theory dealing with additive, multiplicative and fuzzy pairwise comparison matrices which will be used later on. The matter of this section is motivated primarily on [8–10], or [3].

An *abelian group* is a set, G, together with an operation $\odot$ (read: operation *odot*) that combines any two elements $a, b \in G$ to form another element denoted by $a \odot b$. The symbol $\odot$ is a general placeholder for a concretely given operation. The set and operation, $(G, \odot)$, satisfies the following requirements known as the *abelian group axioms*:

- If $a, b \in G$, then $a \odot b \in G$ (*closure*).
- If $a, b, c \in G$, then $(a \odot b) \odot c = a \odot (b \odot c)$ (*associativity*).
- There exists an element $e \in G$ called the *identity element*, such that for all $a \in G, e \odot a = a \odot e = a$ (*identity element*).
- If $a \in G$, then there exists an element $a^{(-1)} \in G$ called the *inverse element to a* such that $a \odot a^{(-1)} = a^{(-1)} \odot a = e$ (*inverse element*).
- If $a, b \in G$, then $a \odot b = b \odot a$ (*commutativity*).

The *inverse operation* $\div$ to $\odot$ is defined for all $a, b \in G$ as follows

$$a \div b = a \odot b^{(-1)}. \tag{2}$$