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Generalized interval vector spaces and interval optimization

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ABSTRACT

This paper presents a method for endowing the generalized interval space with some different structures, such as vector spaces, order relations and an algebraic calculus. With these concepts we formulate interval optimization problems and relate them to classic multi-objective optimization problems. We also present a version of the Von Neumann's Mini-max Theorem in the interval context.

1. Introduction

It is known that digital computers have a limited numerical representation capability. For example, for numbers of kind π and $\sqrt{2}$, that have infinite digits in its decimal places, computer work with rounded floating point numbers. Moreover, most rationals like $\frac{1}{3}$ have rounded representation. Algebraic operations on floating points numbers may have accumulated errors that may be significant. For example in the Gulf War, after the launching of a missile against the U.S. military, a U.S. Patriot missile failed to intercept this attacking missile owing to errors generated by approximations in numbers that were part of algorithm implemented in the Patriot. The result of this was that twenty-eight people died and ninety-eight were injured (see http://www.diale.org/patriot.html for more information).

A way to work with this type of error is to better understand real interval spaces. Any number can be bounded exactly by two adjacent numbers since that $\mathbb{R}$, endowed with the usual structure of order, is an unbounded totally ordered set. When doing mathematical analysis with intervals, one comes face-to-face with the algebraic and analytical structures of spaces associated with intervals.

There are three people who, in the decade of the 50^{ties}, independently developed interval arithmetic, Warmus [20], Sunaga [19], and Moore [13]. Moore, however, is the prime mover of interval analysis and considered the father of its development. His book on interval analysis was published in 1966 [14]. That is, while Warmus and Sunaga were the firsts to create interval arithmetic, the development of mathematics analysis on intervals is a result of the work of Moore, his colleagues, and subsequent researchers.

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One natural application of interval analysis is classical optimization which was studied from the beginning of interval analysis (see [3,5,14,15]) and continues to these days (see [1]). For numerical verification of bounds on global optimization, interval methods are essential. A more recent application of interval analysis is in solving optimization problems that have some inaccurate coefficients arising from rounded values and/or from incomplete information. It is very natural to use the interval analysis to work with this kind of problem because we can transform it into an interval optimization problem.

In the literature we can find articles [2,8,22,23] about interval optimization which relates the interval optimization problems with the multi-objective optimization problems using a particular injection function between the interval space and a Banach space, and/or using a particular order relation. But, when we use a particular injection function and a particular relation of order, we obtain a particular class of problems. In this work, we present an approach that allows us to obtain an equivalence between interval optimization problems and vector optimization problems by making use of a class of bijections, and a generic order relation instead of using a particular injection and a particular order relation. As a result, we unify the theory that appears in the earlier literature and provide results for a larger class of interval optimization problems.

The setting of our work is in a real interval space. It is known that the real interval space, denoted by

$$I(\mathbb{R}) = \{[a_1, a_2] : a_1 \leqslant a_2 \text{ and } a_1, a_2 \in \mathbb{R}\},$$

endowed with the arithmetic developed by Moore, is not a vector space because not every element in this space has additive inverse (see [15]). Thus, all the tools given by classical functional analysis cannot be used.

Some authors, such as [11,12,16,17], work in a "bigger" space than $I(\mathbb{R})$, in the sense of containment. This bigger space is defined by

$$M := I(\mathbb{R}) \bigcup \overline{I(\mathbb{R})},$$

where $\overline{I(\mathbb{R})} = \{[a_1, a_2] : [a_2, a_1] \in I(\mathbb{R})\}$, so that, by equipping M with an arithmetic, we get a new space whose algebraic structure is a vector space.

Because most of interval optimization problems involves partial order relations and the classic interval space is not a vector space, there exists a great difficulty in dealing with optimization problems, let alone minmax type of problems. Here we present, as an application of our approach, an interval version of Von Neumann's Minmax Theorem. In order to present that theorem, we use, indirectly the space M and an specific partial order relation based in the length of the intervals to define the concepts of maximum and minimum of an interval-valued function.

This exposition calls the elements in $I(\mathbb{R})$ *proper intervals* and the elements in $\overline{I(\mathbb{R})}$ *improper intervals*. Since M is formed by proper and improper intervals, M endowed with an algebraic structure of a vector space we call a*1-Dimensional Generalized Interval Vector Space*. The set $M^n = M \times M \times \cdots \times M$ with n-factors, endowed with an algebraic structure of a vector space, is call a *n-Dimensional Generalized Interval Vector Space*.

This work is organized as follows. Section 2 presents a method to obtain various distinct vector space structures in M^n, a class of orders that can be defined in M^n, and some examples of specific ordered vector spaces that have been cited in some of the previous published research. We analyze some properties in these ordered spaces. In Section 3 we present a class of optimization problems that involve n-dimensional interval-valued functions. We show how to find solutions to these problems by means of multi-objective optimization methods and present some examples of specific optimization problems. Section 3 allows us to deal with qualitative and quantitative information in the same problem, since that we can define different kinds of bijections for each interval coordinate. Usually, in the literature it has been treated either qualitative or quantitative information. Finally, in Section 4 we present a version of the Von Neumann's Mini-max Theorem for interval-valued functions.

2. *n*-Dimensional Generalized Interval Vector Spaces

This section presents a way to equip the set M^n with a vector space structure by considering the Euclidean vector space $(\mathbb{R}^{2n}, +, \cdot)$, equipped with the usual operations, and using a bijection between the sets M^n and $\mathbb{R}^{2n}$. We also consider an order relation on $\mathbb{R}^{2n}$ and induce an order relation in M^n by using the bijection between these sets. We also discuss some examples that have been used in other papers as particular cases of our method to obtain the structure of vector space and order relation in interval spaces.

2.1. Generalized interval vector spaces

We begin this subsection with some examples of bijective functions $\varphi : M^n \to \mathbb{R}^{2n}$ that will be important in the development of this article.

Example 2.1. A basic bijective function between M and $\mathbb{R}^2$ is given by

$$\psi([a_1, a_2]) = (\lambda_1 \cdot a_1 + \lambda_2 \cdot a_2, \beta_1 \cdot a_1 + \beta_2 \cdot a_2), \tag{1}$$

where $\lambda_1, \lambda_2, \beta_1, \beta_2 \in \mathbb{R}$ and