Information Sciences 288 (2014) 15–26

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Dissimilarity functions and divergence measures between fuzzy sets

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ARTICLE INFO

Article history:
Received 28 December 2012
Received in revised form 4 July 2014
Accepted 30 July 2014
Available online 7 August 2014

Keywords:
Divergence measure
Dissimilarity function
Fuzzy equivalence
Measure space
Fuzzy set

ABSTRACT

In this paper we propose two approaches to constructing divergence measures. The construction is based on the use of dissimilarity functions and fuzzy equivalencies. Firstly, we introduce some ways of generating dissimilarity functions. Then, we present several formulae of divergence measures. Finally, we examine the properties of divergence measures as a whole.

1. Introduction

In many fields of artificial intelligence, we need to compare the descriptions of objects. This comparison is frequently achieved through a measure intended to determine to which extent the descriptions have common points or differ from each other. In some occasions the comparison of fuzzy sets is done by quantifying the degree of equality or similarity between them, but in other cases we need to compare fuzzy sets by quantifying the degree of inequality or difference between them [1,3,8,10,12,16–18,20,24,25,27].

Several ways of measuring the difference between fuzzy sets by means of some functions have been proposed. It should be mentioned here that in some references these functions are called distance measures [3,8,12,20,22], in some references these functions are called dissimilarity measures [4,10], but in other references these functions are called divergence measures [1,9,15,24,25]. The first axiomatic definition for distance measure was proposed by Liu in [22], where four axioms for distance measure were given. On the basis of exponential operation, Fan and Xie [12] defined a new distance measure. Bloch [3] classified distance measures into four categories through their construction approaches. Almost all the distance measures proposed in the literature can be classified into these four categories. In [4], Bouchon-Meunier et al. defined an M-measure of dissimilarity between fuzzy sets. This dissimilarity measure is indeed a distance measure in the sense of Liu [22] in case it is symmetrical and satisfies the triangular inequality. Based on logarithm operation, Bhandari and Pal [2] defined a divergence measure between fuzzy sets. In [25], Montes et al. introduced a new axiomatic definition for divergence measure. Their approach is based on three axioms modeling the minimal requirements for a function that tries to measure the separation

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http://dx.doi.org/10.1016/j.ins.2014.07.052
0020-0255/

or difference between fuzzy sets. In this paper, we call all functions measuring the difference between fuzzy sets divergence measures.

Based on the above-mentioned definitions of divergence measure, it is possible to define measures of divergence between fuzzy sets and, at the present time, the applications of these measures to the theory of fuzzy questionnaires are being studied. When measuring the divergence degree between fuzzy sets, different divergence measures will obtain different results, which result is the best choice of reflecting the intuitive difference between fuzzy sets should be considered. Thus, it is necessary to find some ways to construct more divergence measures and then compare them with other proposals in the literature. Montes et al. [24,25] proposed a way of constructing divergence measure. This kind of divergence measure is also a divergence measure in the sense of Liu [22], based on fuzzy union and intersection. Bustince et al. [8] normalized the definition of divergence measure in the sense of Liu and presented a method to construct normal divergence measures. Both of Montes' and Bustince's approaches to constructing divergence measures are based on the use of some binary functions with special properties. Bustince et al. [8] named this kind of function restricted dissimilarity function. Although Montes' and Bustince's approaches can be used to construct numerous divergence measures, including many existing ones, there still exist some kinds of divergence measures that cannot be constructed by their approaches (e.g., divergence measures based on the set theoretic approach [3]).

Since dissimilarity function plays an important role in constructing divergence measures, our initial aim in this paper is to present some ways of generating dissimilarity functions. Then, we propose two approaches to constructing divergence measure in the sense of Liu and that in the sense of Montes. The construction is based on the use of dissimilarity functions and fuzzy equivalencies. We have considered the form of divergence measure not only in the case where the universal set is finite but also in the case where the universal set is a measure space. Finally, we examine the properties of divergence measures as a whole.

The rest of the paper is organized as follows. Section 2 reviews some preliminary definitions and results. Section 3 focuses on some ways of generating dissimilarity functions. Section 4 presents several formulae of divergence measures which are composed by dissimilarity functions and fuzzy equivalencies. Section 5 examines some properties of divergence measures. Concluding remarks are contained in Section 6.

2. Preliminaries

In this section we briefly recall, without proof, some preliminary definitions and results. They will be necessary along the other sections of the paper.

Throughout this paper, X is the universal set; $F(X)$ is the class of all fuzzy sets of X; $A(x)$ is the membership function of $A \in F(X)$; $P(X)$ is the class of all crisp sets of X; $A^c \in F(X)$ is the complement of $A \in F(X)$ with $A^c(x) = 1 - A(x)$ for all $x \in X$.

Definition 1 [19]. If a decreasing function $n : [0,1] \to [0,1]$ satisfies the boundary conditions $n(0) = 1$ and $n(1) = 0$, then n is called a fuzzy negation.

Definition 2 [5]. A continuous, strictly increasing function $\varphi : [a,b] \to [a,b]$ with boundary conditions $\varphi(a) = a,\ \varphi(b) = b$ is called an automorphism of the interval $[a,b]$.

Definition 3 [19]. An associative, commutative and increasing function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it has the neutral element equal to 1.

An associative, commutative and increasing function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it has the neutral element equal to 0.

Fodor and Roubens [13] defined fuzzy equivalence as a binary operation on the unit interval in the following way.

Definition 4 [13]. A function $E : [0,1]^2 \to [0,1]$ is called a fuzzy equivalence if it satisfies the following properties:

(E1) $E(x,y) = E(y,x)$ for all $x,y \in [0,1]$.
(E2) $E(x,x) = 1$ for all $x \in [0,1]$.
(E3) $E(1,0) = 0$.
(E4) For all $x,y,x',y' \in [0,1]$, if $x \leqslant x' \leqslant y' \leqslant y$, then $E(x,y) \leqslant E(x',y')$.

It can be proved that (E4) is equivalent to: For all $x,y,z \in [0,1]$, if $x \leqslant y \leqslant z$, then $\min(E(x,y),E(y,z)) \geqslant E(x,z)$.

The following reasonable properties can be considered for fuzzy equivalencies: For all $x,y \in [0,1]$,

(E5) $E(x,y) = 1 \iff x = y$.
(E6) $E(x,y) = 0 \iff \min(x,y) = 0$ and $\max(x,y) \neq 0$.
(E7) $E(x,1) = x$.
(E8) $E(x,y) = E(1-x, 1-y)$.
(E9) $E(x, 1-x) = 0 \iff x = 0$ or $x = 1$.