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Relations among convergence concepts of uncertain sequences

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A B S T R A C T

Uncertainty theory is a generalization of probability theory and credibility theory. The purpose of this paper is to investigate various convergence concepts of uncertain sequences in uncertainty spaces. Sufficient and necessary conditions for convergence of uncertain sequences are given. The relations among various convergence concepts of uncertain sequences are established.

1. Introduction

There are many phenomena in the real world or experiments in many areas of studies whose outcomes cannot be predicted in advance. A lot of effort has been made to address this kind of phenomena. Mathematical structures have been constructed and theories based on these structures have been developed. Among them are probability theory, fuzzy theory [15], credibility theory, and uncertainty theory. One of the most important concepts in the study of these phenomena is the measure of the likelihood of an event. Probability is such a measure in probability theory in which randomness is systematically studied. Possibility measure in fuzzy theory was proposed by Zadeh in 1978 [16] for a fuzzy event and further studied by many researchers including Cooman [1]. Possibility measure lacks the so-called self-duality property. Including the self-duality property in the definition, Liu and Liu [6] introduced credibility measure in 2002. Since then, credibility theory has received considerable attention [3,4,7,8,13].

Uncertainty theory differs from probability theory, fuzzy theory, and credibility theory by its mathematical structure. It is used to model human uncertainty. Some mathematical concepts and theories have been extended to include uncertainty, in other word these concepts and theories are studied in uncertainty spaces. For example, Liu [10] studied uncertain programming to solve optimization problems in uncertain environments such as system reliability design, facility location problem, vehicle routing problem, and project scheduling problem. In [11], Liu proposed uncertain inference including an inference rule. Li and Liu [5] presented uncertain logic in which the truth value is defined as an uncertain measure. Uncertainty theory has become a new tool to describe subjective uncertainty and has wide applications in information sciences. For a detailed discussion on uncertainty theory and its applications, the interested reader may consult the book [12].

As in classical measure theory, probability theory and credibility theory, sequence convergence plays a very important role in uncertainty theory. In fact, Liu [9] introduced various convergence concepts of uncertain variable sequences including convergence almost surely, convergence in measure, and convergence in mean. Recently, You [14] gave the definition of convergence uniformly almost surely and presented some relations among these convergences. Gao [2] gave some mathematical

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properties of uncertain measure. Inspired by these papers, the present paper is devoted to a further discussion on the convergence of uncertain variable sequences.

The paper is organized as follows: In Section 2 we recall some definitions and present some results about uncertain sequences which will be used in the sequel. In Section 3, some necessary and sufficient conditions of convergent sequences are given and the relations among various convergences of uncertain sequences are established. Finally, we give our concluding remarks in Section 4.

2. Preliminaries

Throughout this paper, Γ is a nonempty set and Σ is a σ-algebra over Γ. (Γ,Σ) is called a measurable space. Each element $A \in \Sigma$ is called an event or a measurable set. The complement of an event A is denoted by A^c.

Definition 2.1 [9]. Let (Γ,Σ) be a measurable space. If a real valued function M from Σ to [0,1] satisfies

(1) (Normality) $M(\Gamma) = 1$.
(2) (Monotonicity) if $A_1 \subset A_2$, then $M(A_1) \leqslant M(A_2)$.
(3) (Self-duality) $M(A) + M(A^c) = 1$ for any event A.
(4) (Countable subadditivity) for every countable sequence of events $\{A_i\}$.

$$M\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} M(A_i),$$

then M is called an uncertain measure and (Γ,Σ,M) is called an uncertain space.

Remark 2.1. In Definition 2.1, the monotonicity axiom can be deduced from the rest of axioms, see [12].

Clearly, if (4) is replaced by the following condition (4′) (or (4″)), then M becomes a credibility measure [7] (or a probability measure) and (Γ,Σ,M) becomes a credibility space (or a probability space).

(4′) (Maximality) for every countable sequence of events $\{A_i\}$ with $\sup_i M(A_i) < 0.5$,

$$M\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_i M(A_i).$$

(4″) (Countable Additivity) for every countable sequence of events $\{A_i\}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$M\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} M(A_i).$$

Thus, both credibility measures and probability measures are instances of uncertain measures. However, the converse is not true. The following example shows that an uncertain measure may not be a probability measure or a credibility measure.

Example 2.1. Let $\Gamma = \{x_1,x_2,x_3\}$, $\Sigma = 2^{\Gamma}$. Define $M(\{x_1\}) = 0.5$, $M(\{x_2\}) = 0.4$, $M(\{x_3\}) = 0.3$, $M(\{x_1, x_2\}) = 0.7$, $M(\{x_1,x_3\}) = 0.6$, $M(\{x_2,x_3\}) = 0.5$, $M(\varphi) = 0$, and $M(\Gamma) = 1$. It is not difficult to check that M is neither a credibility measure nor a probability measure, while M is an uncertain measure.

Theorem 2.1. *Let M be an uncertain measure and $\{A_i\}$ be a sequence of events with $M(A_i) \to 0$ as $i \to \infty$. Then for any event A, we have*

$$\lim_{i\to\infty} M(A \cup A_i) = \lim_{i\to\infty} M(A \setminus A_i) = M(A).$$

Proof. By (2) and (4) in Definition 2.1, we have, for each i,

$$M(A) \leqslant M(A \cup A_i) \leqslant M(A) + M(A_i),$$

which leads to that $M(A \cup A_i) \to M(A)$ as $i \to \infty$. Again by (2) and (4) in Definition 2.1 and the fact that $A\backslash A_i \subset A \subset (A\backslash A_i) \cup A_i$, we have

$$M(A\backslash A_i) \leqslant M(A) \leqslant M(A\backslash A_i) + M(A_i),$$

which implies that

$$0 \leqslant M(A) - M(A\backslash A_i) \leqslant M(A_i).$$