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On fuzzy implications determined by aggregation operators

Yao Ouyang

Faculty of Science, Huzhou Teachers College, Huzhou, Zhejiang 313000, People's Republic of China

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ABSTRACT

Fuzzy implication operators play important roles in both theoretical and applied aspects of fuzzy sets theory. Many papers investigated various properties of different types of implications and the interrelationships among these properties. In this paper, we exploit the minimal conditions which must be satisfied for a binary operation A to generate a residual implication with additional properties. It includes several examples to clarify the situation.

1. Introduction

In fuzzy logic systems [15,18,27], connectives "and", "or" and "not" are usually modeled by t-norms, t-conorms, and strong negations on [0, 1] (see, for example [2,17]), respectively. Based on these logical operators on [0, 1], the three fundamental classes of fuzzy implications on [0, 1] (that is, *R*-, *S*-, and *QL*-implications on [0, 1]) were defined and extensively studied (see [3,6,7,19,25] and the references therein).

But, as was pointed out by Fodor and Keresztfalvi [14], sometimes there is no need of the commutativity or associativity for the connectives "and" and "or":

When one works with binary conjunctions and there is no need to extend them for three or more arguments, as happens, e.g. in the inference pattern called generalized modus ponens (GMP for short), associativity of the conjunction is an unnecessarily restrictive condition. The same is valid for the commutativity property if the two arguments have different semantical backgrounds and it makes no sense to interchange one with the other...

Thus, implications based on some other operators like weak t-norms [12] and pseudo t-norms [29] were defined and extensively investigated by many authors.

When investigating the properties of an *R*-implication like operator I, most authors always assumed that it based upon a special class of operators such as t-norms and uninorms [1,5,9,19,23,28]. Let A: $[0, 1]^2 \to [0, 1]$ be an arbitrary binary operator. If we define a residual implication like operator I based upon A, then it would be an interesting topic to investigate the relationship between the properties of I and those of A.

In this paper we focus on this issue (notice that Demirli and De Baets [8] also discussed this problem, see also Jayaram and Mesiar [16]). The paper is organized as follows. Section 2 reviews some basic concepts and notions which will be used in the paper. Section 3, the main part of this paper, explores the *R*-implication like operators while Section 4 investigates the *S*-implication like operators, including a lot of interesting examples. Section 5 summarizes our results.

E-mail address: oyy@hutc.zj.cn

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2. Preliminaries

In this section, we review some basic concepts which will be used in the paper.

Definition 2.1 ([2,17]). A triangular norm is a binary operation T: $[0,1]^2 \to [0,1]$, such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$ (commutativity);
(T2) $T(T(x,y),z) = T(x,T(y,z))$ (associativity);
(T3) $T(x,y) \leqslant T(x,z)$ whenever $y \leqslant z$ (monotonicity);
(T4) $T(x,1) = x$ (boundary condition).

Definition 2.2 ([2,17,24]). A t-norm T is said to be

(i) continuous if it is continuous in each arguments;
(ii) left-continuous if it is a left-continuous two-place function;
(iii) Archimedean if for every $x,y \in (0,1)$ there is $n \in N$ such that $x_T^{(n)} < y$, where $x_T^{(1)} = x$ and $x_T^{(n+1)} = T(x_T^{(n)}, x)$ for $n \geqslant 1$;
(iv) strict if T is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$;
(v) nilpotent if T is continuous and if each $x \in (0,1)$ is a nilpotent element, i.e., if for each $x \in (0,1)$ there exists $n \in N$ such that $x_T^{(n)} = 0$.

The most important nilpotent t-norm is the Łukasiewicz t-norm T_L defined by $T_L(x,y) = \max(x+y-1,0)$, and the most important strict t-norm is the product $T_P(x,y) = xy$ while the most popular continuous non-Archimedean t-norm is the minimum $T_M(x,y) = \min(x,y)$. Notice that the commutativity (T1), the monotonicity (T3) as well as the boundary condition (T4) imply that every t-norm is bounded from above by minimum. If we replace the boundary condition (T4) of a t-norm by a weak condition $T(x,y) \leqslant T_M(x,y)$, then we obtain a triangular subnorm (t-subnorm, for short) [21].

Definition 2.3 ([2]). A triangular conorm (for short, t-conorm) is a binary operation S: $[0,1]^2 \to [0,1]$, such that for all $x,y \in [0,1]$,

$$S(x,y) = 1 - T(1-x, 1-y) \tag{2.1}$$

holds for some given t-norm T.

Notice that if (2.1) holds then we say T and S are dual to each other. A t-conorm S is said to be continuous (Archimedean, strict, nilpotent, respectively) if its dual t-norm T is continuous (Archimedean, strict, nilpotent, respectively). The most important nilpotent t-conorm is the Łukasiewicz t-conorm S_L defined by $S_L(x,y) = \min(x+y,1)$, and the most important strict t-conorm is the probabilistic sum $S_P(x,y) = x+y-xy$ while the most popular continuous non-Archimedean t-conorm is the maximum $S_M(x,y) = \max(x,y)$.

Definition 2.4 ([10]). A binary operator T: $[0,1]^2 \to [0,1]$ is called a semicopula if it satisfies

(i) $T(x,1) = T(1,x) = x$, $\forall x \in [0,1]$;
(ii) $\forall x_1, x_2, y_1, y_2$ in $[0,1]$, if $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ then $T(x_1,y_1) \leqslant T(x_2,y_2)$.

It should be pointed out that the semicopula was introduced by Suárez García and Gil Álvarez [26] (under the name of t-seminorm) to define integrals. Notice that some authors introduced other binary operators to define implications, among which we mention Fodor's weak t-norms and Wang and Yu's pseudo-t-norms.

Definition 2.5 ([12]). A function T: $[0,1]^2 \to [0,1]$ is called a weak t-norm if it satisfies the following conditions:

(i) $T(a,1) \leqslant a$ and $T(1,b) = b$ for any $a, b \in [0,1]$;
(ii) $T(a,b) \leqslant T(c,d)$ whenever $a \leqslant c$ and $b \leqslant d$.

Definition 2.6 ([29]). A function T: $[0,1]^2 \to [0,1]$ is called a pseudo-t-norm if it satisfies the following conditions:

(i) $T(1,a) = a$ and $T(0,b) = 0$ for any $a, b \in [0,1]$;
(ii) $T(a,b) \leqslant T(a,c)$ whenever $b \leqslant c$.

Notice that a pseudo-t-norm is just a binary operation which is non-decreasing in the second argument, has a left neutral element and satisfies $T(0,1) = 0$. Clearly, any t-norm is a semicopula, any semicopula is a weak t-norm and any weak t-norm is a pseudo-t-norm.