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Three-way decisions space and three-way decisions

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ABSTRACT

Ideas of three-way decisions proposed by Yao come from rough sets. It is well known that there are three basic elements in three-way decisions theory, which are ordered set as to define *three regions*, object set contained in evaluation function and evaluation function to make three-way decisions. In this paper these three basic elements are called decision measurement, decision condition and evaluation function, respectively. In connection with the three basic elements this paper completes three aspects of work. The first one is to introduce axiomatic definitions for decision measurement, decision condition and evaluation function; the second is to establish three-way decisions space; and the third is to give a variety of three-way decisions on three-way decisions spaces. Existing three-way decisions are the special examples of three-way decisions spaces defined in this paper, such as three-way decisions based on fuzzy sets, random sets and rough sets etc. At the same time, multi-granulation three-way decisions space and its corresponding multi-granulation three-way decisions are also established. Finally this paper introduces novel dynamic two-way decisions and dynamic three-way decisions based on three-way decisions spaces and three-way decisions with a pair of evaluation functions.

1. Introduction

Theory of three-way decisions (3WD) is an extension of classic two-way decisions (2WD) [50,51,55,56], whose basic ideas come from Pawlak rough sets [34,35] and probability rough sets [7,19,29,51–58] and whose main purpose is to interpret the positive, negative and boundary regions of rough sets as three decisions outcomes, acceptance, rejection, and uncertainty (or deferment) in a ternary classification respectively. In addition to rough sets as delegates of the three-way decisions, there are other uncertainties, such as fuzzy uncertainty and random uncertainty. Table 1.1 lists typical delegates of three-way decisions.

It can be shown that, under certain conditions, probabilistic three-way decisions are superior to both Palwak three-way decisions and two-way (i.e., binary) decisions [56]. Many recent studies further investigated extensions and applications of three-way decisions [21,26–28,46,51,55,56]. The researches on three-way decisions mainly focus on the following two aspects.

- The first aspect is the background researches on three-way decisions. It mainly contains the extension researches of rough sets. The first class is the extension from Pawlak rough sets to probability rough sets, such as decision-theoretic rough sets (DTRS) [7,55–58], variable precision rough sets (VPRS) [18,62], Bayesian rough sets (BRS) [42], game-theoretic rough sets

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Table 1.1
Representatives of three-way decisions.

Three-way decisions	Acceptance	Rejection	Uncertainty
Rough sets	Positive region	Negative region	Boundary region
Fuzzy sets	Belong to (kernel)	Do not belong to (non-support set)	Uncertain (boundary)
Random sets	Necessary event	Impossible event	Uncertain event

(GTRS) [13], fuzzy rough sets/rough fuzzy sets (FRS/RFS) [9], interval-valued fuzzy rough sets (IVFRS) [12,15,43] and Dominance-based fuzzy rough sets [8,10]. The second class is the extension from one granular to multi-grualation, such as multi-granulation rough sets (MGRS) [38–40], multi-granulation decision-theoretic rough sets [41], multi-granulation rough sets based covering [23], Neighborhood-based multi-granulation rough sets (NMGRS) [24] and so on.

- The second aspect is theoretical framework researches on three-way decisions. It mainly contains value domain of evaluation functions [51], construction and interpretation of evaluation functions [51,55,56] and the mode of tree-way decisions [51].

These researches, however, are premature in theory and there are some problems about three-way decisions.

The first one is a measurement problem on decision conclusions (decision domain). More popular now is a linear order or totally ordered set whose typical representative is [0, 1]. The so-called linear order set $(L,\preceq)$ means that $\preceq$ denotes a linear order relation (or total order relation) on L, i.e., it satisfies the following conditions.

(1) Reflexivity $x \preceq x$.
(2) Anti-symmetry $x \preceq y,\ y \preceq x \Rightarrow x = y$.
(3) Transitivity $x \preceq y,\ y \preceq z \Rightarrow x \preceq z$.
(4) Comparability $x, y \in L \Rightarrow x \preceq y$ *or* $y \preceq x$.

But some problems may not be solved by a linear order for decision-making. Yao in [51] used a partially ordered set L and divided L into two nonempty sets, i.e. $L = L^- \cup L^+$ $(L^- \cap L^+ = \emptyset)$, where L^- is used to reject and L^+ is used to accept. Indeed the problems of three-way decisions come down to the partitions, but he did not give any methods to divide. This paper uses a complete distributive lattice with an inverse order and involutive operator (for short, fuzzy lattice) as a measurement tool, so that its applications are much more comprehensive.

The second one is a decision condition problem (decision condition domain). Current common conditions used to three-way decisions are subsets of universe, fuzzy sets [59] or shadowed sets [36,37]. It is unified to mappings from universe to fuzzy lattice in this paper.

The third one is an evaluation function problem. Evaluation functions are a key to decision-making. Different evaluation functions determine different decision results. Popular evaluation functions are associated with the conditional probability formulae. For example, in probabilistic rough sets [58], there are many models such as decision-theoretic rough sets based on Bayesian risk analysis (DTRS) [17,26,27,51–58], variable precision rough sets (VPRS) [18,62], Bayesian rough sets (BRS) [42,61,63] and fuzzy probabilistic rough sets [16,22,45]. In these models, probability Pr $(C|[x]_R)$ or $\frac{|X\cap[x]_R|}{|[x]_R|}$ (R is an equivalence relation) or $\Pr(X \mid [x]_{R_\lambda})$ (R is a fuzzy equivalence relation) are used as evaluation functions. This paper unifies the evaluation functions through their common properties.

The rest of this paper is organized as follows. In Section 2, the measures of three-way decisions are specified in fuzzy lattice represented by [0, 1], evaluation function axioms are given and three-way decisions spaces are established. Section 3 establishes three-way decisions theory on three-way decisions space, which contains general three-way decisions, lower and upper approximations induced by three-way decisions and multi-granulation three-way decisions. In Section 4, the existing three-way decisions come down to special cases of three-way decisions spaces, including three-way decisions based on fuzzy sets, interval-valued fuzzy sets, shadowed sets, interval sets, random sets and probability sets. At the same time, their corresponding multi-granulation three-way decisions are introduced. Section 5 gives novel dynamic two-way decisions and dynamic three-way decisions based on three-way decisions spaces. Section 6 presents three-way decisions with a pair of evaluation functions based on three-way decisions spaces. Finally this paper is concluded and two questions are discussed.

2. Three-way decisions space

In this section three-way decisions space (3WDS) is established through unifying decision measurement, decision conditions and evaluation functions of three-way decisions.

2.1. Measurement of three-way decisions

In [0, 1], operator $x^c = 1 - x$ $(x \in [0, 1])$ is applied. Similar operator can be discussed in general partially ordered set. For the convenience of research, the concept of involutive negator is defined as follows.