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Prominent classes of the most general subsumptive solutions of Boolean equations

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ABSTRACT

In a recent seminal paper, Rudeanu derived necessary and sufficient conditions for the most general form of the subsumptive general solution of a Boolean equation. The present paper investigates four prominent classes of the aforementioned most general form. These include the conventional (eliminants) class, the don't-care class, the Rudeanu class, and the complete-sum (CS) class. We show that each of these classes satisfies the Rudeanu conditions, and hence fits into the frame of the most general form of the subsumptive solution. The four classes are of variable but comparable computational complexities. The conventional (eliminants) class is the least likely to produce a compact solution, while the don't-care class is the most likely to produce such a solution. Three examples are presented to demonstrate solutions produced by the four classes and compare them form the point of view of compactness of solutions, which determines the ease of generating the tree of particular solutions. A remaining open question that has yet to be settled is whether these four classes are exhaustive, or in other words, whether there are general solution classes other than them. We demonstrate that the don't-care class is never inferior (and occasionally superior) to the other classes. In fact, this class seeks global minimality and has a better control on the pertinent variables and algebra generators. Both the CS class and the Rudeanu class have an advantage over the eliminants method. The CS class utilizes a canonical representation that explicitly shows all information in the simplest form. The Rudeanu class enhances the eliminants class by incorporating the consensus (Boolean resolution) concept in a way similar to that of the CS approach.

1. Introduction

The study of Boolean equations in arbitrary Boolean algebras continued in the past two centuries with fundamental contributions from prominent scholars such as Boole, Schröder, Whitehead, Poretski, Löwenheim, Rudeanu, and Brown [7,8,35,41,44]. The importance of Boolean-equation solving can hardly be overestimated. It permeates many subareas of digital or logical design, such as decomposition of Boolean functions, fault diagnosis and hazard-free synthesis of digital circuits, construction of binary codes, flip–flop excitation, and the design of sequential circuits [7,8]. Other areas of modern science to which Boolean-equation solving has been applied include biology, grammars, chemistry, law, medicine, spectroscopy, and graph theory [7,8]. In many of these areas, the use of arbitrary or "big" Boolean algebras is unavoidable and definitely useful, even if unrecognizable [7,8]. Many important problems in operations research can be reduced to the problem of solving a

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system of Boolean equations. The solutions of Boolean equations serve also as an important tool in the treatment of pseudo-Boolean equations and inequalities, and their associated problems in integer linear programming [13,20,21].

To solve a system of Boolean equations, the equations are usually combined into an equivalent single Boolean equation whose set of solutions is exactly the same as that of the original system of equations. This is conceptually simpler than obtaining the set of solutions for each equation and then forming the intersection of such sets to obtain the set of solutions of the overall system. Typically, general subsumptive solutions (wherein each of the variables is expressed as an interval based on successive conjunctive or disjunctive eliminants of the original function) or general parametric solutions (wherein each of the variables is expressed via arbitrary parameters, i.e., freely chosen elements of the underlying Boolean algebra) are sought, from which an exhaustive enumeration of particular solutions can be readily obtained [2,7,8,12,20,23–25,35,41,42,44,47,49–51,55–57,63,64]. By a particular solution we mean an assignment from the underlying Boolean algebra to every pertinent variable that makes the Boolean equation an identity.

In a recent seminal paper, Rudeanu [44] presents a deep, elegant, and concise theorem that is a culmination of almost half a century dedicated to serious study of Boolean equations (see, e.g., [20,35,41]). This theorem states the necessary and sufficient conditions for the most general form of the subsumptive general solution of a Boolean equation. The inherent beauty of this theorem could be appreciated to varying degrees not only by first-class mathematicians but also by other mathematically-mature scientists.

In the present paper, we try to shed some light on the Rudeanu Theorem in [44] and explore further its practical utility. We hope to pave the way for researchers in the areas of logic, artificial intelligence, digital design, and operations research to derive direct benefit from this theorem. In particular, we stress that any existing or forthcoming subsumptive class of solutions for Boolean equations can be validated via the conditions required by Rudeanu Theorem. On the other hand, an erroneous or fallacious class of solutions would be outright rejected on the grounds that it fails to satisfy these conditions.

We note that the only demonstration for the utility of Rudeanu Theorem in [44] was to show that the classical subsumptive general solution, based on conjunctive eliminants, satisfies the necessary and sufficient conditions required by the theorem. We extend and supplement this demonstration by proving that three other important classes of solutions also fit into the most general form of the general subsumptive solution, as each of them satisfies the necessary and sufficient conditions of Rudeanu. The first class is a class of solutions due to Brown [7,8] and Rushdi [47,49] that uses partially-defined functions of conjunctive eliminants. The second class is an algebraic class of solutions proposed by Rudeanu [42]. The third class is a novel class of solutions derived from the complete sum (CS) or Blake canonical form (BCF) [1,6–8,10,11,14–16,19,22,27,28,31,33,35,41,48,51–53,59,60,62]. For convenience, we will refer to these three classes of solutions as the don't-care solutions, the Rudeanu solutions, and the CS solutions, respectively. Naming one class of solutions after Rudeanu stresses the fact that credit for this class goes solely to him, and does not undermine his contributions to other classes. We demonstrate the four prominent classes of solutions with three illustrative examples. We also compare them from the point of view of compactness of solutions which determines the ease of generating the tree of particular solutions.

The rest of this paper is organized as follows. Section 2 reviews the eliminants and don't-care subsumptive solutions of an arbitrary Boolean equation of one unknown, while Section 3 does the same for an arbitrary Boolean equation of several unknowns. Section 4 briefly states the necessary and sufficient conditions set by Rudeanu [44] for the most general form of subsumptive general solutions. Sections 5 and 6 prove that each of the don't-care and Rudeanu solutions satisfies the necessary and sufficient conditions of Rudeanu, and hence fits into the framework of the most general subsumptive solutions. Section 7 introduces the CS subsumptive general solution and proves that it satisfies the Rudeanu conditions. Section 8 discusses relations among the four classes of solutions, demonstrates and compares them via three illustrative examples, and finally poses an open question concerning the existence of other classes of general subsumptive solutions. Section 9 concludes the paper.

2. Solution of an arbitrary Boolean equation of one unknown

This section reviews the theory for finding the subsumptive solution of an arbitrary Boolean equation in one unknown over a general Boolean algebra **B**. More details are available in [2,7,8,24,25,35,41,44,47,49,63,64]. In a Boolean equation of one unknown

$$f(X) = 0, \tag{1}$$

where $f(X)$: $\boldsymbol{B} \to \boldsymbol{B}$, the function $f(X)$ can be replaced according to the Boole–Shannon expansion [7,8,35,41,45,46,53] by

$$f(X) = f(0)\overline{X} \vee f(1)X. \tag{2}$$

In (2), the two terms $(f(0)\overline{X})$ and $(f(1)X)$ have a single opposition (due to the variable X that appears complemented in the former term and un-complemented in the latter), and hence they have a consensus [6–8,10,11,15,16,19,27,33,35,41,48,52,53,62] with respect to X, namely $(f(0)\,f(1))$. The expression for $f(X)$ in (2) can be augmented by this consensus to become

$$f(X) = f(0)\overline{X} \vee f(1)X \vee f(0)f(1). \tag{3}$$

The Eq. (1) is satisfied if and only if each term in (3) is equal to 0, viz.,