Information Sciences 281 (2014) 338–354

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Characterization and reduction of concept lattices through matroid theory

Hua Mao *

Department of Mathematics, Hebei University, Baoding 071002, China

ARTICLE INFO

Article history:
Received 22 November 2012
Received in revised form 13 May 2014
Accepted 20 May 2014
Available online 2 June 2014

Keywords:
Matroidal space
Concept lattice
Reduction
Attribute
Object
A-bi-circuit

ABSTRACT

Concept lattice theory is an efficient tool for data analysis. Reduction of attributes and objects for a context is an important issue in concept lattice theory. We explore the relationships between matroidal spaces and concepts for a given context. With the assistance of matroid theory, we conduct the attribute reduction and object reduction for a context and formulate the concepts for a given context. As a generalization of the linear independence in vector spaces, matroids provide well established platforms for some algorithms such as greedy algorithm.

In this paper, we mainly propose a notion of matroidal space by the family of circuits of a matroid. These relationships characterize the constructions of not only concepts, but also concept lattices. Meanwhile, we demonstrate that reduction of attributes and objects proposed in this paper preserve the original hierarchy order in the concept lattice for a given context.

1. Introduction

The theory of concept lattices, a method of data analysis, was first proposed by Wille [22] and further embellished by Carpineto and Romano [3], Ganter et al. [4,5] and Wei et al. [19]. A concept lattice is an ordered hierarchy that is defined by a binary relationship between objects and attributes in a data set, and as such, incorporates certain general and specific concepts. As an efficient tool of data analysis and knowledge processing, the concept lattice theory has been applied to fields such as knowledge engineering, data mining, information searches, and software engineering (see Carpineto and Romano [3] and Ganter et al. [4]).

Gross data reduction is needed for knowledge discovery and data mining and, towards this end, concept lattice analysis has achieved recent and abundant results in both these fields. Reducing the size of many concept lattices is necessary to gain useful results. Ganter and Wille [5] proposed reducible attributes and reducible objects from the viewpoint of shortening lines or rows. By searching for a minimal subset of attributes so that the new concept lattice is isomorphism with the raw concept lattice, Zhang et al. [23] presented another way to reduce attributes. Relationships between reducts and concept lattices are discussed also such as [4,5,19,23]. Based on the above Ganter et al. and Zhang et al. for reducts of concept analysis respectively, there are many other methods to reduce the size of the concept lattices using various mathematical theories such as fuzzy sets [7,9], Boolean algebra [13], Galois connection [10], matrix [16], and rough sets [1,8].

---

* Tel.: +86 3125079351.
E-mail address: yushengmao@263.net

http://dx.doi.org/10.1016/j.ins.2014.05.044

After analyzing the methods in [5,7–10,13,16,23], we find that every method of them comes from the combination of a mathematical theory and concept lattices. Hence, if we hope to develop the reducts of concept lattices, we should explore a mathematical structure, which is different from the known in the study on the reducts of concept lattices, to apply in the research. To pursuit this aim, we notice the following points (1.1) and (1.2):

(1.1) In combinatorics, a branch of mathematics, matroid theory was introduced by Whitney [21]. This theory provides a unifying abstract treatment of dependence in linear algebra and graph theory. Matroids also play a fundamental role in combinatorial optimization and are widely used in greedy algorithms (see Oxley [14] and Welsh [20]). Mao [11] deals with the relationships between matroids and concept lattices. This same author utilizes some properties of concept lattices to research the axioms of matroids [12].

(1.2) Pawlak [15] presents the definition of an ‘information system’. Comparing his definition to Ganter and Wille’s [5] definition of a ‘context’, we find that a context is a particular information system. Tang et al. [17] and Wang et al. [18] studied information systems and both used attribute reduction with matroid theory. We can see that many algorithms favored for the structures of concept lattices and attribute reductions have the same general ideas associated with greedy algorithms. These results imply that matroid theory may enlighten us on the use of attribute reduction when addressing concept lattices.

The above (1.1) and (1.2) shows that the use of matroids to achieve context reduction, based on attribute or object reduction, would be very useful in the study of concept lattices. To date however, this research topic is unexplored. If we could use matroids to reduce concept lattices, then the corresponding results between knowledge reduction in concept lattices and matroid theory could also be interchangeable. We can confirm that this interchangeability will be good for both concept lattices and matroids, and equally useful in searching methods of concepts in a context.

In this paper, we attempt to make a contribution to the study of concept lattices by using matroids. As we know, it is somewhat difficult to understand matroids directly. And this difficulty also arises when we consider the combination of matroids and concept lattices. So, in order to give an intuitive interpretation to this combination, we will first study it from the connection between matroids and contexts. By applying the axioms of circuits of matroids, we present the notion of a-bi-circuit and formulate the concepts in a context based on this notion. Next, we provide the notion of matroidal space based on the circuit axioms of a matroid and the notion of concepts for a context. Additionally, with the assistance of matroidal spaces, we characterize every concept in a context. Finally, for the two definitions of attribute reduction, as per Ganter and Zhang respectively, we characterize attribute reduction and object reduction based on matroidal spaces.

The rest of this paper is organized in five sections as follows. In Section 2, we review some basic facts relative to concept lattices and matroids. In Section 3, we discuss the relationships between matroids and concept lattices and provide a notion of matroidal space. Using these relationships, with the assistance of matroidal spaces, we present some characterizations of concept lattices. In Section 4, according to the definition of reduction in a context cf. Ganter and Zhang respectively, we deal with the reduction of concept lattices with matroidal spaces. The final Section 5, concludes the paper.

We declare that in this paper,

(1) all the discussions are finite;
(2) all the notions and properties for lattice theory are referred to Birkhoff [2] and Grätzer [6];
(3) if two lattices L_1 and L_2 are isomorphic, then it is in notation $L_1 \cong L_2$.

2. Preliminaries

As stated above, this section introduces some basic facts of concept lattices and matroids.

2.1. Concept lattice

This subsection gives only a brief overview of the basic notations and terminologies for concept lattices. For a more detailed description, please refer to [5].

Definition 2.1.1

(1) [5, p.17]. A *context* $\mathbb{K} := (O, P, I)$ consists of two sets O and P and a relation I between O and P. The elements of O are called the *objects* and the elements of P are called the *attributes* of the context. The relation I is also called the *incidence relation* of $\mathbb{K}$. We write oIp or $(o,p) \in I$.

(2) [5, p.18]. For a set A of objects, we define $A' := \{p \in P | oIp \text{ for all } o \in A\}$. Correspondingly, for a set B of attributes, we define $B' := \{o \in O | oIp \text{ for all } p \in B\}$.
A *concept* of a context (O, P, I) is a pair (A, B) with $A \subseteq O, B \subseteq P, A' = B$ and $B' = A$. We call A the *extent* and B the *intent* of the concept (A, B).