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Indicators for the characterization of discrete Choquet integrals

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Ordered weighted averaging (OWA) operators and their extensions are powerful tools used in numerous decision-making problems. This class of operator belongs to a more general family of aggregation operators, where each operator of this family is understood as a discrete Choquet integral. Aggregation operators are usually characterized by indicators. In this article four indicators usually associated with the OWA operator are extended to the discrete Choquet integral: namely, the degree of balance, the divergence, the variance indicator and Rényi entropies. All of these indicators are considered from a local and a global perspective. Linearity of indicators for linear combinations of capacities is investigated and, to illustrate the usefulness of results, indicators of the probabilistic ordered weighted averaging (POWA) operator are derived. Finally, a numerical example is provided to discuss the results in a specific context.

1. Introduction

Aggregation operators are very useful tools for summarizing information and have been widely used in recent decades [1,13,37]. In this context, the Choquet integral [6], a class of integral linked to non-additive measures, has taken a leading role. Integrals are used to aggregate values of functions, and as such can be understood as aggregation operators. The Choquet integral includes a wide range of aggregation operators as particular cases. Over the last few years, the Choquet integral has received much attention from researchers, and this has generated new extensions and generalizations of this class of integral. For instance, Greco et al. [15] proposed an extension of the Choquet integral in which the capacity depends on the values to be aggregated. Similarly, Yager [45] presented new induced aggregation operators inspired by the Choquet integral and Xu [40] introduced some intuitionistic fuzzy aggregation functions also based on the Choquet integral. Klement et al. [17] presented a universal integral that covers the Choquet and the Sugeno integral for non-negative functions, while Torra and Narukawa [38] studied a generalization of the Choquet integral inspired by the Losonczi mean. Bolton et al. [5] connected the Choquet integral with distance metrics and, more recently, Torra and Narukawa [39] introduced an operator that generalizes the Choquet integral and the Mahalanobis distance.

Two particular cases of aggregation operators that can be understood as a Choquet integral are the weighted arithmetic mean (WAM) and the ordered weighted averaging (OWA) operator [41]. Several authors have turned their attention to the study of the OWA operator [47], since it serves to provide a parameterized family of aggregation operators between the min-

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imum and the maximum. In order to assess OWA operators appropriately, indicators for characterizing the weighting vector are required. Initially, Yager [41] introduced the orness/andness indicators and the entropy of dispersion for just this purpose. Later, he proposed complementary indicators, including the balance indicator [42] and the divergence [44], to be used in exceptional situations. Meanwhile, Fullér and Majlender [11] suggested the use of a variance indicator and Majlender [21] introduced the Rényi entropy [32] as a generalization of the Shannon entropy [33] in the framework of the OWA operator. Some of these indicators have been extended for the Choquet integral. For example, Marichal [24] Grabisch et al. [13] presented several types of degree of orness indicators: Marichal [24] for the Choquet integral, while Grabisch et al. [13] for general aggregation functions. Likewise, Yager [43] Marichal [23] and Kojadinovic et al. [18] studied the entropy of dispersion in the framework of the Choquet integral. Marichal and Roubens [22] analyzed the relationship between the alternative definitions of the entropy of dispersion indicator introduced by Yager [43] and Marichal [23]. However, to the best of our knowledge, additional indicators have yet to be defined for the Choquet integral.

The aim of this article is to further enrich the present set of indicators for the Choquet integral, by incorporating new ones to earlier contributions and by presenting an unified compilation of indicators for describing its aggregation features. Four indicators commonly used for the OWA operator – that is, the degree of balance, the divergence, the variance indicator and Rényi entropies- are extended to the discrete Choquet integral. The advantage of incorporating these additional indicators is that they can help to cover a wide range of situations, including exceptional types of aggregation that cannot be correctly identified by means of the degree of orness or the entropy of dispersion. Two different perspectives are considered so as to allow both local and global indicators to be defined.

The linearity of indicators is investigated when dealing with linear combinations of capacities. Indicators are presented for the probabilistic OWA (POWA) operator [26,29], which deals with a linear combination of two particular cases of the Choquet integral (the OWA and the WAM) in order to obtain more complex aggregations. The importance of these two aggregation operators is determined by the particular weight assigned to them in the linear combination.

Finally, an example is presented to numerically illustrate our results in a specific context, namely a hypothetical customer on-line satisfaction assessment conducted using survey analysis and a set of particular cases of the Choquet integral. The main advantage of using the Choquet integral is that a wide range of scenarios and attitudes can be considered and the one in closest accordance with our interests can then be selected. The example includes the estimation of indicators that identify different features linked to each considered Choquet integral.

The rest of this paper is organized as follows. In Section 2 some basic preliminaries regarding the OWA operator and the Choquet integral are briefly reviewed. In Section 3 the main indicators for characterizing the OWA operator and the existing indicators for the Choquet integral are compiled. New indicators for the Choquet integral, the degree of balance, the divergence, the variance indicator and Rényi entropies are presented in Section 4. A concise analysis of the linearity of indicators with respect to linear combinations of capacities is presented in Section 5. In addition, the indicators inherited by the POWA operator, understood as a Choquet integral, are also provided in this section. In Section 6 an illustrative example is given and in Section 7 the main conclusions of the article are summarized.

2. OWA operators and the Choquet integral

The relationship between the OWA operator and the discrete Choquet integral is described in this section. Henceforth, let $N = \{m_1, \ldots, m_n\}$ be a finite set and $2^N = \wp(N)$ be the set of all subsets of N.

2.1. OWA operators

Ordered weighted averaging (OWA) operators were first introduced by Yager [41]. Let $\vec{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ be such that $\sum_{i=1}^n w_i = 1$. The OWA operator with respect to $\vec{w}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $\mathrm{OWA}_{\vec{w}}(x_1, x_2, \ldots, x_n) := \sum_{i=1}^n x_{\sigma(i)} \cdot w_i$, where σ is a permutation of $(1, 2, \ldots, n)$ such that $x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \ldots \leqslant x_{\sigma(n)}$, i.e. $x_{\sigma(i)}$ is the ith smallest value of $x_1, x_2, \ldots, x_n$.

OWA operators are inspired by the concept of the weighted arithmetic mean (WAM) but requiring the components of $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ to be ordered before the aggregation is computed.[2] For convenience, we consider the components of $\vec{x}$ in ascending as opposed to descending order. The OWA operator has been widely developed in the literature [47]. For example, Yager [46] proposed the use of generalized means to extend the OWA operator. A further interesting generalization of the OWA operator is the Quasi-OWA operator, in which quasi-arithmetic means[3] are used [10]. A Quasi-OWA operator is defined by $\mathrm{Quasi-OWA}_{\vec{w}}(x_1, x_2, \ldots, x_n) := g^{-1}\left(\sum_{i=1}^n g\left(x_{\sigma(i)}\right) \cdot w_i\right)$, where $g : \mathbb{R} \to \mathbb{R}$ is a strictly continuous monotonic function.

2.2. The Choquet integral

In order to analyze the Choquet integral the concept of capacity must first be defined. A capacity or a fuzzy measure on N *is a mapping from* 2^N *to* $[0, K_\mu]$ *for some* $K_\mu > 0$, which satisfies that $\mu(\emptyset) = 0$ and that if $A \subseteq B$ then $\mu(A) \leqslant \mu(B)$, for any $A, B \in 2^N$ (monotonicity).

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[2] Note that the WAM with respect to $\vec{w}$ is an aggregation operator defined as $\mathrm{WAM}_{\vec{w}}(x_1, x_2, \ldots, x_n) := \sum_{i=1}^n x_i \cdot w_i$. It is an aggregation operator on $\mathbb{R}^n$.

[3] Merigó and Gil-Lafuente [25] presented similar generalizations when dealing with induced aggregation operators.