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Two new characterizations of universal integrals on the scale $[0,1]$

Salvatore Greco [a,b], Radko Mesiar [c,d], Fabio Rindone [a,*]

[a] Department of Economics and Business, 95029 Catania, Italy
[b] Portsmouth Business School, Operations & Systems Management University of Portsmouth, Portsmouth PO1 3DE, United Kingdom
[c] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
[d] Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic

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The concept of universal integral, recently proposed and axiomatized, encompasses several integrals, including the Choquet, Shilkret and Sugeno integrals. In this paper we present two new axiomatizations of universal integrals on the scale $[0,1]$. In the first characterization, we look at universal integrals on the scale $[0,1]$ as families of aggregation functions $\mathcal{F}$ satisfying some desired properties. The second characterization is given in the framing in which the original definition of universal integral was provided.

1. Introduction

Non-additive integrals are the integrals that are based on monotone (non-necessarily additive) measures. In the last decades the use of non-additive integrals has become pervasive in Decision Analysis. For example in the field of multiple-criteria decision aid (MCDA) (see [6] for a survey on MCDA) the Choquet integral [4] and the Sugeno integral [24], have become useful tools to represent interaction of criteria [8,10].

Also in decision making under risk and uncertainty the Expected Utility Theory (EUT) of von Neumann and Morgenstern [27], based on (additive) Lebesgue integral, has revealed to be inadequate to explain human behavior in many situations (see e.g. [1,5,14,25]). For this motivation, more general theories, called non-EUT theories have been developed (for a seminal survey we recommend [22]). Non-EUT theories are often based on non-additive integrals. For example, in decision making under risk and uncertainty, the Choquet integral has firstly received an axiomatic characterization [19] and then has been successfully applied to economic models of decision, like the Choquet Expected Utility (CEU) of Schmeidler and Gilboa [7,20] and the Cumulative Prospect Theory of Kahneman and Tversky [26]. Very recently, the bipolar Choquet integral of Grabisch and Lebreuche [9] has been applied in order to obtain a generalization of the CPT which does not imply gain–loss separability [13].

Klement et al. have recently proposed the concept of universal integral [17]. The family of universal integrals contain several well known non-additive integrals, like the Choquet integral [4], the Sugeno integral [24] and the Shilkret integral [21]. A further generalization is represented by the family of universal integrals computed with respect to a level dependent capacity [15,16]. A level dependent capacity depends also on the value of the aggregated variables and can be expressed by means of a system of capacities (see [12] for further details). Again, this concept generalizes several previous definitions, like the

* Corresponding author. Tel.: +39 3486988832.
E-mail addresses: salgreco@unict.it (S. Greco), radko.mesiar@stuba.sk (R. Mesiar), frindone@unict.it (F. Rindone).

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level-dependent Choquet integral [12], the level-dependent Shilkret integral [3] and the level-dependent Sugeno integral [18].

Any kind of integrals can be seen as a family of functionals with special properties. In the framework of MCDA these functionals turn to be special aggregation functions. In this paper we present two new characterizations of universal integrals on the scale $[0,1]$. In the first characterization we start by assuming a family of aggregation functions is given. We demonstrate that when the aggregation functions in the family satisfy a set of desired properties, then the family can be seen as a universal integral. As a consequence, we elicit a second axiomatization of universal integrals on the scale $[0,1]$ in the original setting proposed in [17]. We provide also some illustrative examples.

The paper is organized as follows. In Section 2 we recall the definition of universal integral, and in Section 3 we concentrate on universal integrals on the scale $[0,1]$. Section 4 shows how universal integrals can be characterized in terms of families of aggregation functions satisfying a set of given properties. In Section 5 we elicit a new characterization of universal integrals on the scale $[0,1]$ in the original setting. In Section 6, we present conclusions.

2. Universal integrals

A measurable space $(X,\mathcal{A})$ is a nonempty set X equipped with a σ-algebra $\mathcal{A}$. Given a measurable space $(X,\mathcal{A})$, a function $f: X \to [0,\infty]$ is $\mathcal{A}$-measurable if, for each $B \in \mathcal{B}([0,\infty])$, the σ-algebra of Borel subsets of $[0,\infty]$, the preimage $f^{-1}(B)$ is an element of $\mathcal{A}$.

For each $A \subseteq X$ we denote with $\mathbf{1}_A$ the function on X defined by: $\mathbf{1}_A(x) = 1$ if $x \in A$, $\mathbf{1}_A(x) = 0$ else.

A *monotone measure* on a measurable space $(X,\mathcal{A})$ is a function $m: \mathcal{A} \to [0,\infty]$ satisfying the following conditions:

1. *Boundary conditions*: $m(\emptyset) = 0$ and $m(X) > 0$.
2. *Monotonicity*: $m(A) \leqslant m(B)$ for all $A, B \in \mathcal{A}$ such that $A \subseteq B$.

A monotone measure m satisfying $m(X) = 1$ is also called *capacity* or *fuzzy measure* [24,4].

Let $(X,\mathcal{A})$ be a measurable space. We shall use the following notations:

1. $\mathcal{F}^{(X,\mathcal{A})}$ denotes the set of all $\mathcal{A}$-measurable functions $f: X \to [0,\infty]$; for all $f \in \mathcal{F}^{(X,\mathcal{A})}$, the level set $\{x \in X \mid f(x) \geqslant t\}$, $t \in [0,1]$, is briefly denoted by $\{f \geqslant t\}$.
2. For each $a \in]0,\infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ denotes the set of all monotone measures on $(X,\mathcal{A})$ satisfying $m(X) = a$, and we put
$$\mathcal{M}^{(X,\mathcal{A})} = \cup_{a\in]0,\infty]}\mathcal{M}_a^{(X,\mathcal{A})}.$$
3. Let $\mathcal{S}$ be the class of all measurable spaces, and put
$$\mathcal{D}_{[0,\infty]} = \cup_{(X,\mathcal{A})\in\mathcal{S}}\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}.$$

Definition 1. A pseudo-multiplication is a function $\otimes: [0,\infty]^2 \to [0,\infty]$ such that for all x, y, t and $z \in [0,\infty]$ the following properties are satisfied:

- monotonicity: $x \otimes y \leqslant t \otimes z$, whenever $x \leqslant t$ and $y \leqslant z$;
- zero is an annihilator: $x \otimes 0 = 0 \otimes x = 0$;
- neutral element: there exists $e \in]0,\infty]$ such that $e \otimes x = x \otimes e = x$.

In [17] the following definition has been given.

Definition 2. A function $\mathbf{I}: \mathcal{D}_{[0,\infty]} \to [0,\infty]$ is called a universal integral, if the following axioms hold:

(I1) for any measurable space $(X,\mathcal{A})$, the restriction of $\mathbf{I}$ to $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ is nondecreasing in each coordinate;

(I2) there exists a pseudo-multiplication $\otimes: [0,\infty]^2 \to [0,\infty]$ such that for all $(X,\mathcal{A}) \in \mathcal{S}$, $m \in \mathcal{M}^{(X,\mathcal{A})}$, $A \in \mathcal{A}$ and $c \in [0,\infty]$,
$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A);$$

(I3) $\mathbf{I}(m_1,f_1) = \mathbf{I}(m_2,f_2)$ for all $(m_1,f_1), (m_2,f_2) \in \mathcal{D}_{[0,\infty]}$ such that
$$m_1(\{f_1 \geqslant t\}) = m_2(\{f_2 \geqslant t\}), \quad \text{for all } t \in]0,\infty].$$

For each pair $(m,f) \in \mathcal{D}_{[0,\infty]}$, consider the function $h^{(m,f)}:]0,\infty] \to [0,\infty]$ defined by $h^{(m,f)}(t) = m(\{f \geqslant t\})$, for all $t \in]0,\infty]$. For each $(m,f) \in \mathcal{D}_{[0,\infty]}$, $h^{(m,f)}$ is nonincreasing and Borel measurable. Observe that the function $h^{(m,f)}$ can be seen as a generalized survival function (dual to the distribution function for a random variable), and then the axiom (I3) expresses a generalization of a well known fact from the probability theory, namely, that two random variables possessing the same distribution function have the same expected value. Similarly, axiom (I2) can be seen as a generalization of the fact that if a random variable V has as its range $\{0,c\}$ for some constant c, then its expected value depends only on c and $P(\{V = c\})$.