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Soft subsets and soft product operations

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ABSTRACT

Molodtsov's soft set theory provides a general mathematical framework for dealing with uncertainty. It is known that soft subsets and soft equal relations are of vital importance in soft set theory. This paper aims to give a systematic study on several types of soft subsets and various soft equal relations induced by them. We give some equivalent characterizations of different soft subsets and endeavor to ascertain the interrelations among these notions, illustrated by a number of concrete examples. We also consider ontology-based soft sets and show that soft L-subsets generalize both soft M-subsets and ontology-based soft subsets.

Moreover, by means of soft L-subsets and some related notions, we give a theoretical study concerning soft product operations such as $\wedge$-products and $\vee$-products. We consummate some incomplete results concerning soft product operations existing in the literature, and investigate the algebraic properties of soft product operations in detail. Finally, we consider free soft algebras associated with soft product operations. It is shown that soft L-equal relations are congruence relations over free soft algebras and the corresponding quotient structures form commutative semigroups.

1. Introduction

In modern time, the mathematical modelling and manipulating of various types of uncertainties has become an increasingly important issue in solving complicated problems arising in a wide range of areas such as engineering, economics, environmental science, medicine and social sciences. Although probability theory, fuzzy set theory [32], rough set theory [27] and interval mathematics [12] are well-known and effective tools for coping with vagueness and uncertainty, each of them has certain inherent limitations; one major weakness shared by these mathematical methodologies may be due to the inadequacy of parametrization tools [26].

In 1999, Molodtsov [26] proposed *soft set theory*, as a generic mathematical approach to vagueness and uncertainty, which is free from the difficulty affecting the above-mentioned methods. Since then there has been a rapid growth of interest in soft sets and their various applications to algebraic systems [1,6,15–19,33], ontology [13], data analysis [34], forecasting [30], simulation [21], and decision making under uncertainty [29,4,5,7,8]. It is interesting to see that soft sets are closely related to many other soft computing models such as rough sets and fuzzy sets. Many researchers contributed to extending soft sets with fuzzy set theory [23,25,31]. Feng et al. [9] combined soft sets with rough sets and fuzzy sets, obtaining three types of hybrid models: rough soft sets, soft rough sets, and soft-rough fuzzy sets. Using soft sets as the granulation structures called soft approximation spaces, soft rough approximations and soft rough sets were introduced, which can be seen as the

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generalization of Pawlak's rough set model based on soft set theory [10]. Ali [3] discussed the fuzzy sets and fuzzy soft sets induced by soft sets. To extend the expressive power of soft sets, Jiang et al. [13] presented ontology-based soft sets, in which the concepts of description logics act as the parameters of soft sets. Note that ontology-based soft sets was successfully applied to semantic decision making problems involving user queries [14].

There is no doubt that soft subsets and soft equal relations play a fundamental role in soft set theory. Maji et al. [24] first defined the notion of soft subsets in a very strict manner (see Definition 2.3 in [24]). They presented some results on soft distributive laws of soft product operations (see Proposition 2.6 in [24]) without the mathematical proof of validity. However, Ali et al. [2] pointed out that Maji's soft distributive laws do not hold with respect to the soft equal relation given by Maji et al. [24] (see Remark 2.8 in [2]). Moreover, Feng et al. [9] gave a new definition of soft subsets which can be seen as a generalization of Maji's soft subsets. Qin and Hong [28] also introduced two new types of soft equal relations which are congruence relations on soft sets. To investigate basic properties of ontology-based soft sets, Jiang et al. [13] also defined the notion of logical subsets and added some new feature to Maji's soft subsets by means of description logics. Recently, in the study of interval-valued fuzzy soft sets, Jun and Yang [20] considered a generalization of soft subsets and tried to amend Maji's soft distributive laws using generalized soft equal relations. They presented a latest result called generalized soft distributive laws of soft product operations.

Inspired by Jun and Yang's novel ideas, Liu, Feng and Jun further defined soft L-subsets and soft L-equal relations in a short research note [22]. An important result obtained there indicated that distributive laws do not hold with respect to all kinds of soft equal relations existing in the literature. The present study is a continuation of the above line of research, particularly what has been initiated in [22]. In order to make the paper self-contained, we would like to briefly recall some important concepts and results in [22] (although this might cause some overlap). Compared with the note [22], this subsequent work mainly concentrates on five different types of soft subsets and algebraic properties of soft product operations. Beyond distributive laws which have been extensively studied by many researchers, we shall also consider commutative laws, associative laws, idempotent laws and other fundamental properties. A more interesting thing is that all these naturally give rise to a type of quotient structures of free soft algebras associated with soft product operations. In addition, we also consider ontology-based soft sets and try to give some motivation of soft subsets from ontology viewpoint.

The remainder of this paper is organized as follows. Section 2 recalls some fundamental concepts such as Molodtsov's soft sets, soft M-subsets and soft F-subsets. Section 3 is a brief introduction to ontology-based soft sets and soft subsets proposed by Jiang et al. [13]. In Section 4 we discuss two kinds of generalized soft subsets including soft J-subsets and soft L-subsets. Section 5 is devoted to a detailed study on soft product operations and their algebraic properties with respect to various soft subsets and soft equal relations. Section 6 establishes the quotient structures of free soft algebras associated with soft product operations. Finally, conclusions are presented in the last section.

2. Soft sets and soft subsets

Let U be the so-called universe of discourse and E be the universe of all possible parameters related to the objects in U. The pair (U,E) is called a *soft universe*. Here we assume that both U and E are nonempty finite sets.

Definition 2.1 [24]. Let $\mathcal{P}(U)$ denote the power set of U. A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U, where $A \subseteq E$ and $F: A \to \mathcal{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set $\mathfrak{S}$.

By definition, a soft set $\mathfrak{S} = (F, A)$ over U can be viewed as a parameterized family of subsets of the universe U. For any parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ may be interpreted as the set of ϵ-*approximate elements* [26]. It is worth noting that $F(\epsilon)$ may be arbitrary: some of them may be empty, and some may have nonempty intersections [26]. In what follows, we shall always consider soft sets with nonempty parameter sets in the soft universe (U,E) unless otherwise stated. That is, when we talk about a soft set (F,A) over U, we always assume that $\emptyset \neq A \subseteq E$.

Maji et al. [24] initiated the concept of soft subsets and soft equal relations (hereinafter we called soft M-subsets and soft M-equal relations) in the following manner:

Definition 2.2 [24]. Let (F,A) and (G,B) be two soft sets over U. Then (F,A) is called a *soft M-subset* of (G,B), denoted $(F, A) \widetilde{\subseteq}_M (G, B)$, if $A \subseteq B$ and $F(a) = G(a)$ (i.e., $F(a)$ and $G(a)$ are identical approximations) for all $a \in A$. Two soft sets (F,A) and (G,B) are said to be *soft M-equal*, denoted $(F, A) =_M (G, B)$, if $(F, A) \widetilde{\subseteq}_M (G, B)$ and $(G, B) \widetilde{\subseteq}_M (F, A)$.

We give below a simple characterization of Maji's soft subsets and soft equal relations.

Proposition 2.3. *Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (G, B)$ be two soft sets over U. Then we have*

(1) $\mathfrak{S} \widetilde{\subseteq}_M \mathfrak{T}$ *if and only if $A \subseteq B$ and $G|_A = F$ (i.e., the approximate function of $\mathfrak{S}$ coincides with that of $\mathfrak{T}$ restricted to A).*
(2) $\mathfrak{S} =_M \mathfrak{T}$ *if and only if $A = B$ and $G = F$.*

Proof. The proof is straightforward and thus omitted. □

Note that there exist different definitions of soft subsets and soft equal relations as follows (hereinafter we called soft F-subsets and soft F-equal relations).