Information Sciences 278 (2014) 883–899

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

The structures of intuitionistic fuzzy equivalence relations

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ARTICLE INFO

Article history:
Received 7 August 2013
Received in revised form 19 February 2014
Accepted 19 March 2014
Available online 1 April 2014

Keywords:
Intuitionistic fuzzy equivalence relations
α-Cut relations
Chain structure
Partially ordered structure

ABSTRACT

In this paper, we investigate some structures of intuitionistic fuzzy equivalence relations. We show that the family of all intuitionistic fuzzy equivalence relations on the set X is a complete lattice. Unlike the chain structure of the cut relations of fuzzy equivalence relations, there are two ordered structures concerned with the α-cut relations of intuitionistic fuzzy equivalence relations. We investigate the chain structure and the partially ordered structure of α-cut relations of intuitionistic fuzzy equivalence relations. We also show that the intuitionistic fuzzy equivalence relations can be applied for clustering analysis.

1. Introduction

As a generalization of fuzzy sets [30], the concept of intuitionistic fuzzy sets (IFSs) was introduced by Atanassov [1,3]. Atanassov also originally introduced intuitionistic fuzzy relations (IFRs) [2,5], which were recently suggested to be called Atanassov's intuitionistic fuzzy relations or just bipolar fuzzy relations [17,18] or bifuzzy relations [20]. This change is due to a terminological difference between Atanassov's intuitionistic fuzzy relations (and sets) and what is currently understood as intuitionistic logic [17]. But, Atanassov [7] pointed out that there is still not a feasible alternative which is accepted by the fuzzy set community. In this paper, we are not involved in this discussion and still use the terms in the sense of Atanassov's.

In classical set theory, a relation from a set X to a set Y is formally defined as a subset of the Cartesian product $X \times Y$, accordingly, a fuzzy relation R from a set X to a set Y is defined as a fuzzy set in the Cartesian product $X \times Y$, and an IFR R from a set X to a set Y is defined as an IFS in the Cartesian product $X \times Y$. For x in X and y in Y, $R(x,y)$ represents the degree to which x stands in relation R with y. A (fuzzy, or intuitionistic fuzzy) relation from X to X is simply called a (fuzzy, or intuitionistic fuzzy) relation on X.

An important relation is the equivalence relation. The fuzzy equivalence relation (FER) is the generalization of the equivalence relation under fuzzy environment, and the intuitionistic fuzzy equivalence relation (IFER) is the generalization of the equivalence relation under intuitionistic fuzzy environment. We denote $\mathbb{ER}(X)$ as the family of equivalence relations on X, denote $\mathbb{FER}(X)$ as the family of FERs on X, and denote $\mathbb{IFER}(X)$ as the family of IFERs on X. It is well-known that each equivalence relation on X corresponds to a unique subdivision of X. For any two equivalence relations $R_1, R_2 \in \mathbb{ER}(X)$, if $\forall x, y \in X$, $(x,y) \in R_2 \Rightarrow (x,y) \in R_1$, i.e., $R_2 \subseteq R_1$, then we say that R_2 is finer than R_1, and write $R_1 \preccurlyeq R_2$. $\preccurlyeq$ is a partial order on $\mathbb{ER}(X)$. The greatest element in $\mathbb{ER}(X)$ is $\Delta(X) = \{(x,x) | x \in X\}$, the corresponding subdivision of X is $\{\{x\} | x \in X\}$, it is the finest subdivision

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http://dx.doi.org/10.1016/j.ins.2014.03.101
0020-0255/

of X. The least element in $\mathbb{ER}(X)$ is $X \times X = \{(x, y)|x, y \in X\}$, the corresponding subdivision of X is $\{X\}$, it is the coarsest subdivision of X. If R is an FER on X, for any $\lambda_1, \lambda_2 \in [0,1], R_{\lambda_1}$ and R_{λ_2} are the cut relations of R, respectively, then R_{λ_1} and R_{λ_2} are both equivalence relations. It is clear that if $\lambda_1 \leqslant \lambda_2$, then $R_{\lambda_2} \subseteq R_{\lambda_1}$, and R_{λ_2} is finer than R_{λ_1}. In this case, the family of cut relations $\{R_\lambda|\lambda \in [0,1]\}$ is a chain structure. Related work appeared in [33,34], where the authors introduced the theory of fuzzy quotient space and gave several equivalent statements for the FERs.

Fuzzy sets and relations have many applications in diverse types of areas [22], for example in data bases, pattern recognition, neural networks, fuzzy modeling, economy, medicine, and multi-attribute decision making (MADM). Atanassov's intuitionistic fuzzy sets and relations are also widely applied in solving real-life problems. IFRs has been investigated in many papers such as [8–12,14,18,21,23,24,28].

In this paper we mainly investigate the IFERs. There are two main differences between the FERs and the IFERs:

(1) Assume R is a FER and R' is an IFER. The family of cut relations of R has a chain structure, while the family of cut relations of R' has two possible structures, i.e., the chain structure and the partially ordered structure. Moreover, it is disclosed that if the family of cut relations of R' has the chain structure, then we can find a FER R such that R and R' have the same cut relations. If the family of cut relations of R' has partially ordered structure, then we cannot find any FER R such that R and R' have the same cut relations in this case.
(2) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. Assume $R = (R(x_i, x_j))_{n\times n}$ is a fuzzy relation defined on X, and $R' = (R'(x_i, x_j))_{n\times n}$ is an IFR defined on X. Let $V(R) = \{\lambda|\lambda = R(x_i, x_j)$ for some $i, j = 1, 2, \ldots, n\}$ and $V(R') = \{\alpha|\alpha = R'(x_i, x_j)$ for some$i, j = 1, 2, \ldots, n\}$. The elements of $V(R)$ are real numbers in $[0,1]$ while the elements of $V(R')$ are intuitionistic fuzzy values (IFVs). It is well-known that R is a FER on X if and only if the cut relation R_λ is an equivalence relation on X for any $\lambda \in V(R)$. The result cannot be generalized to the intuitionistic fuzzy case. In this paper, we will give a counter example and illustrate that an IFR R' may not be an IFER even if the cut relation R'_α is an equivalence relation on X for each $\alpha \in V(R')$.

With regard to the above two differences, in this paper, we will investigate the IFERs from new viewpoints. We will see that unlike the chain structure of cut relations of FERs, there are two different structures about the α-cut relations of IFER. We will investigate the chain structure and the partially ordered structure of the α-cut relations of IFER.

We organize the remainder of this paper as follows. In Section 2, we give some basic concepts such as IFSs, IFVs, IFRs and IFERs. We also give some related operation laws and results. In Section 3, we investigate the ordered algebra structure of IFERs. In Section 4, we investigate the chain structure and the partially ordered structure of α-cut relations of IFER R. In Section 5, we investigate the least element in $V(R)$. In Section 6, we give a decomposition theorem of IFERs. Section 7 gives some properties of IFERs. In Section 8, the IFERs are applied for clustering analysis. Section 9 provides the concluding remarks.

2. Preliminaries

Let us first recall some basic concepts and results.

A fuzzy relation on a non-empty set X is a mapping $R : X \times X \to I = [0,1]$. If X is a finite set and card$(X) = n, n \in \mathbb{N}$, then a fuzzy relation $R : X \times X \to I$ may be represented by a matrix $R = (R(x_i, x_j))_{n\times n}$.

Let $\mathbb{FR}(X)$ be the family of fuzzy relations on non-empty set X. For any $R \in \mathbb{FR}(X)$ [19]:

(1) If for any $x \in X, R(x, x) = 1$, then R has the reflexive property;
(2) If for any $x \in X, R(x, x) = 0$, then R has the irreflexive property;
(3) If for any $x, y \in X, R(x, y) = R(y, x)$, then R has the symmetric property;
(4) If for any $x, y, z \in X, \sup_{y\in X} \min(R(x, y), R(y, z)) \leqslant R(x, z)$, then R has the transitive property;
(5) If for any $x, y, z \in X, \inf_{y\in X} \max(R(x, y), R(y, z)) \geqslant R(x, z)$, then R has the dual transitive property.

Definition 2.1. An intuitionistic fuzzy set (IFS) A in X is given by Atanassov [3,6]:

$$A = \{(x, \mu_A(x), \nu_A(x))|x \in X\},$$

where $\mu_A : X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$ with the condition $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for each $x \in X$. The values $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A. The family of all IFSs in X will be denoted by $\mathbb{IFS}(X)$.

For each IFS A in X, let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), x \in X$, then $\pi_A(x)$ is the hesitancy degree of x to A. It is obvious that $\pi_A(x) \in [0,1], x \in X$. If $\pi_A(x) = 0$ for every $x \in X$, i.e., $\mu_A(x) = 1 - \nu_A(x)$, then the IFS A reduces to a fuzzy set.

For any given x, the pair $(\mu_A(x), \nu_A(x))$ is called an intuitionistic fuzzy value (IFV) by Xu [27,29]. For convenience, the pair $(\mu_A(x), \nu_A(x))$ is often denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$, where $\mu_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$ and $\mu_\alpha + \nu_\alpha \leqslant 1$. Moreover, it is clear that each IFV $\alpha = (\mu_\alpha, \nu_\alpha)$ corresponds to an interval $[\mu_\alpha, 1 - \nu_\alpha]$. If $\mu_\alpha = 1 - \nu_\alpha$, then the IFV α reduces to a crisp number.