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Maximally local connectivity and connected components of augmented cubes ☆

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ABSTRACT

The *connectivity* of a graph is an important issue in graph theory, and is also one of the most important factors in evaluating the *reliability* and *fault tolerance* of a network. It is known that the *augmented cube* AQ_n is *maximally connected*, i.e. $(2n-1)$-connected, for $n \geqslant 4$. By the classic *Menger's Theorem*, every pair of vertices in AQ_n is connected by $2n-1$ *vertex-disjoint paths* for $n \geqslant 4$. A routing with parallel paths can speed up transfers of large amounts of data and increase fault tolerance. Motivated by research on networks with faults, we obtained the result that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 2n-7$ for $n \geqslant 4$, each pair of non-faulty vertices, denoted by u and v, in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $AQ_n - F$, respectively. Moreover, we demonstrate that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 4n-9$ for $n \geqslant 4$, there exists a large connected component with at least $2^n - |F| - 1$ vertices in $AQ_n - F$, which improves on the results of Ma et al. (2008) who show this for $n \geqslant 6$.

1. Introduction

Interconnection networks have been widely studied recently. The architecture of an interconnection network is usually denoted as an undirected graph G. For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a,b) | (a,b)\ (a \neq b)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. The interconnection network topology is usually represented by a graph $G = (V, E)$, where vertices represent processors and edges represent links between processors. The *neighborhood* of vertex v, denoted by $N(v)$, is $\{x | (v, x) \in E\}$. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of vertices in $N(v)$. A graph G is *k-regular* if $\deg(v) = k$ for every vertex $v \in V$. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in the literature. Graph theory can be used to analyze network reliability, so we use the terminology *graphs* and *networks* synonymously.

The *reliability* and *fault tolerance* of a network with respect to processor failures is directly related to the *connectivity* of the corresponding graph. Connectivity is one of the important factors for evaluating the fault tolerance of a network [3,4,14]. The connectivity of G, written $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and

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$\kappa(G) = |V(G)| - 1$ otherwise. Traditional connectivity only considers how many faulty vertices there can be before the network fails. It is known that $\kappa(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G. For the most part, even if the number of faulty vertices is higher than that specified by network connectivity standards, the network remains connected or at least a large part of it remains connected. Many measures of fault tolerance of networks are related to the maximal size of the connected components of networks with faulty vertices, so it is essential to estimate the maximally connected component of the network with the faulty vertices [1]. Yang et al. [15–17] have proposed a way to determine the maximally connected component of the *n-dimensional hypercube*.

A distributed system is useful because it offers the advantage of improved connectivity. Menger's Theorem [10] shows that if a network G is k-connected, every pair of vertices in G is connected by k vertex-disjoint (parallel) paths. Efficient routing can be achieved using vertex-disjoint paths, providing parallel routing and high fault tolerance, increasing the efficiency of data transmission, and decreasing transmission time. Saad and Schultz [12] studied the n vertex-disjoint parallel paths of an n-dimensional hypercube Q_n. Day and Tripathi [7] discussed the $n-1$ vertex-disjoint parallel paths of an n-dimensional star graph S_n for any two vertices of S_n.

Many useful topologies have been proposed to balance performance and cost parameters. Among them, the binary *hypercube* Q_n [5,12] is one of the most popular topologies, and has been studied for parallel networks. *Augmented cubes* are derivatives of the hypercubes with good geometric features that retain some favorable properties of the hypercubes, such as vertex symmetry, maximum connectivity, best possible wide diameter, routing, and broadcasting procedures with linear time complexity. The augmented cube of dimension n, denoted by AQ_n, is a Cayley graph, $(2n-1)$-regular, $(2n-1)$-connected, and has diameter $\lceil n/2 \rceil$ [6]. In this paper, we demonstrate a tight result that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 2n-7$ for $n \geqslant 4$, each pair of non-faulty vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $AQ_n - F$, respectively. In addition, we consider the maximally connected component of the augmented cube with faulty vertices. In 2008, Ma et al. showed that for $n \geqslant 6$, for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 4n-9$, the maximally connected component of $AQ_n - F$ has at least $2^n - |F| - 1$ vertices. We improve this result by demonstrating it for $n \geqslant 4$.

In the next section, we give the definition of the augmented cube AQ_n for $n \geqslant 1$. Section 3 deals with the maximally connected component of $AQ_n - F$ with $|F| \leqslant 4n-9$ for $n \geqslant 4$. Section 4 studies the vertex-disjoint fault-free paths in $AQ_n - F$ with $|F| \leqslant 2n-7$ for $n \geqslant 4$.

2. The augmented cube AQ_n

The definition of the *n-dimensional augmented cube* is stated as the following. Let $n \geqslant 1$ be a positive integer. The n-dimensional augmented cube [6,8], denoted by AQ_n, is a vertex transitive and $(2n-1)$-regular graph with 2^n vertices. Each vertex is labeled by an n-bit binary string and $V(AQ_n) = \{u_n u_{n-1} \ldots u_1 | u_i \in \{0,1\}\}$. AQ_1 is the complete graph K_2 with vertex set $\{0,1\}$ and edge set $\{(0,1)\}$. As for $n \geqslant 2$, AQ_n consists of (1) two copies of $(n-1)$-dimensional augmented cubes, denoted by AQ^0_{n-1} and AQ^1_{n-1}; and (2) 2^n edges (two perfect matchings of AQ_n) between AQ^0_{n-1} and AQ^1_{n-1}. AQ_n can be written as $AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ for $n \geqslant 2$. $V(AQ^0_{n-1}) = \{0u_{n-1}u_{n-2}\ldots u_1 \mid u_i \in \{0,1\}\}$ and $V(AQ^1_{n-1}) = \{1v_{n-1}v_{n-2}\ldots v_1 \mid v_i \in \{0,1\}\}$. Vertex $u = 0u_{n-1}u_{n-2}\ldots u_1$ of AQ^0_{n-1} is joined to vertex $v = 1v_{n-1}v_{n-2}\ldots v_1$ of AQ^1_{n-1} if and only if either.

(i) $u_i = v_i$ for $1 \leqslant i \leqslant n-1$; in this case, (u, v) is called a *hypercube edge* and we set $v = u^h$, or
(ii) $u_i = \bar{v}_i$ for $1 \leqslant i \leqslant n-1$; in this case, (u, v) is called a *complement edge* and we set $v = u^c$.

The augmented cubes $AQ_1,\ AQ_2$, and AQ_3 are illustrated in Fig. 1. Let the hypercube edge set of AQ_n be E^h_n and the complement edge set of AQ_n be E^c_n. Thus, $E^h_n = \{(u, u^h) | \ u \in V(AQ^0_{n-1})\}$ and $E^c_n = \{(u, u^c) | \ u \in V(AQ^0_{n-1})\}$. Obviously, each of E^h_n and E^c_n is a perfect matching between the vertices of AQ^0_{n-1} and AQ^1_{n-1}. Then, both $|E^h_n|$ and $|E^c_n|$ are equal to 2^{n-1}.

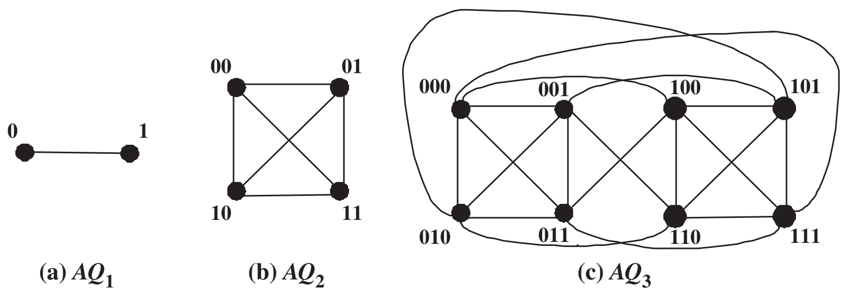

Fig. 1. The augmented cubes $AQ_1,\ AQ_2$, and AQ_3.