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Runtime analysis of a multi-objective evolutionary algorithm for obtaining finite approximations of Pareto fronts

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ABSTRACT

Previous theoretical analyses of evolutionary multi-objective optimization (EMO) mostly focus on obtaining ϵ-approximations of Pareto fronts. However, in practical applications, an appropriate value of ϵ is critical but sometimes, for a multi-objective optimization problem (MOP) with unknown attributes, difficult to determine. In this paper, we propose a new definition for the finite representation of the Pareto front—the adaptive Pareto front, which can automatically accommodate the Pareto front. Accordingly, it is more practical to take the adaptive Pareto front, or its ϵ-approximation (termed the ϵ-adaptive Pareto front) as the goal of an EMO algorithm. We then perform a runtime analysis of a $(\mu+1)$ multi-objective evolutionary algorithm ($(\mu+1)$ MOEA) for three MOPs, including a discrete MOP with a polynomial Pareto front (denoted as a polynomial DMOP), a discrete MOP with an exponential Pareto front (denoted as an exponential DMOP) and a simple continuous two-objective optimization problem (SCTOP). By employing an estimator-based update strategy in the $(\mu+1)$ MOEA, we show that (1) for the polynomial DMOP, the whole Pareto front can be obtained in the expected polynomial runtime by setting the population size μ equal to the number of Pareto vectors; (2) for the exponential DMOP, the expected polynomial runtime can be obtained by keeping μ increasing in the same order as that of the problem size n; and (3) the diversity mechanism guarantees that in the expected polynomial runtime the MOEA can obtain an ϵ-adaptive Pareto front of SCTOP for any given precision ϵ. Theoretical studies and numerical comparisons with NSGA-II demonstrate the efficiency of the proposed MOEA and should be viewed as an important step toward understanding the mechanisms of MOEAs.

1. Introduction

Recently, various soft computing techniques have been widely utilized in the fields of science and engineering [30,37,9,22,36,39]. One set of powerful soft computing method is multi-objective evolutionary algorithms (MOEAs). These algorithms can explore the feasible spaces of multi-objective optimization problems (MOPs) to obtain uniformly distributed Pareto vectors, which has been shown by abundant numerical results [41,24,42,10,43,21,11,2,38,44,7,13,23,29,34,35]. Meanwhile, theoretical studies of convergence [26,25,16,40,32,8,1] and runtime analyses [14,26,28,31,5,6,18,19,3,15,20,4,12,33] have also been performed to explain how MOEAs function on different MOPs.

Laumanns et al. [27,28] investigated the "leading ones, trailing zeros" (LOTZ) problem and demonstrated that the expected runtime of the simple evolutionary multi-objective optimizer (SEMO) for LOTZ is $\Theta(n^3)$. Giel [14] extended the runtime analysis to the Global SEMO (GSEMO) by investigating the LOTZ problem and another simple test problem, and

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Neumann [31] found that the GSEMO can accommodate the Pareto front of a multi-objective minimum-spanning tree problem in the expected pseudo-polynomial runtime if the Pareto front is strongly convex. Moreover, Horoba [20] showed that the diversity-maintaining evolutionary multi-objective optimizer (DEMO) is a fully polynomial-time randomized approximation scheme for multi-objective shortest path problems. To theoretically confirm the efficiencies of hypervolume-based MOEAs, Beume et al. [3] compared the individual-based $\mathcal{S}$ metric selection evolutionary multi-objective optimization algorithm (SMS-EMOA) with the single-individual models of the nondominated sorting genetic algorithm II (NSGA-II) and the improved strength Pareto evolutionary algorithm (SPEA2), and then investigated the convergence rates of several population-based variants of SMS-EMOA [4].

By adding objectives to a well-known plateau function, Brockhoff et al. [6] found that changes in running time are caused by changes in the dominance structure. Subsequently, Schütze et al. [33] demonstrated that even if an increase in the number of objectives makes the problem more difficult, this increase in difficulty is sometimes not significant. Moreover, Laumanns *et al.* [28] verified the population's beneficial function through rigorous runtime analyses, while Giel and Lehre [15] further declared that there could be an exponential runtime gap between the population-based algorithms and single individual-based algorithms.

To understand the convergence properties of population-based MOEAs more concretely, Brockhoff et al. [5] analyzed the hypervolume-based MOEAs and obtained a polynomial upper bound on the expected runtime—to obtain an ϵ-approximation of an exponentially large Pareto front. By analyzing the runtime behaviors of MOEAs employing different diversity-preserving mechanisms, Friedrich et al. [12] demonstrated that certain mechanisms can improve the efficiencies of MOEAs on certain MOPs. Meanwhile, Horoba and Neumann [18,19] proposed several sufficient conditions for obtaining ϵ-Pareto sets of some MOPs they investigated. The theoretical results showed that although an ϵ-dominance approach can help achieve a good approximation for a Pareto set for some MOPs, this approach sometimes prevents the population from distributing uniformly along a small Pareto front. However, an MOEA based on a density estimator performs well in this case.

Existing theoretical results on runtime analysis have generally focused on dominance- or indicator-based MOEAs that were employed to obtain an ϵ-Pareto front of an MOP. To obtain an ϵ-Pareto front, the population size μ must be greater than or equal to a given threshold M, and the case where $\mu < M$ has not yet been considered. For a given precision ϵ, it is hardly feasible to choose a proper population size μ when an MOP with unknown attributes is encountered, whereas a large population will lead to high computation complexity and a small approximate Pareto front cannot represent the whole Pareto front precisely. By incorporating a fitness function compatible with the dominance relation in a $(\mu + 1)$ MOEA, we take a so-called adaptive Pareto front [8] as the destination of population evolution, which can automatically accommodate the true Pareto front. Compared with NSGA-II and SPEA2, the $(\mu + 1)$ MOEA employs a strategy of population update based on a fitness function, by which the selection pressure can be greatly improved when applied to many-objective evolutionary problems. It can also eliminate the essential difficulty of the multi-objective evolutionary algorithm based on decomposition (MOEA/D), that is, the difficulty of generating a uniformly-distributed vector set guiding the evolution of the population. Then, we estimate the expected runtime of a $(\mu + 1)$ MOEA for obtaining adaptive Pareto fronts or ϵ-adaptive Pareto fronts of MOPs. The major contributions of this paper include:

- We take the adaptive Pareto front as the destination of population evolution, and in this way, eliminate the difficult task of selecting a rational population size for a given precision ϵ.
- We theoretically demonstrate that if the $(\mu + 1)$ MOEA is utilized to solve a discrete MOP with polynomial Pareto vectors (the LOTZ), it is more efficient to set the population size equal to the number of Pareto vectors rather than employ a small population to obtain a uniform representation of the Pareto front.
- For a discrete MOP, when the number of Pareto vectors is of exponential order (the LF'_δ), the universal upper bound of the expected runtime is also exponential. However, a polynomial increase in the expected runtime can also be obtained by setting $k - 1 < \frac{n}{\mu} \leqslant k$ for a given positive constant k, where n is the problem size and μ is the population size.
- We demonstrate that a $(\mu + 1)$ MOEA based on a density estimator is a good solver for an MOP with a Pareto front that is a continuous curve because, for any $\varepsilon > 0$, it can obtain an ε-approximation of the adaptive Pareto front in the expected polynomial runtime.
- By comparing a variant of the proposed $(\mu + 1)$ MOEA, termed the $(\mu + \mu)$ MOEA, with NSGA-II, we also show that the proposed method is competitive with some existing MOEAs.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries on MOPs and MOEAs, and in Section 3, we perform the runtime analysis of the proposed $(\mu + 1)$ MOEA for the three MOPs under investigation. To demonstrate the efficiency of the newly proposed MOEA, we compare numerical results with the NSGA-II in Section 4. Finally, Section 5 concludes the paper and presents future work to be carried out.

2. Preliminaries

2.1. Multi-objective optimization problems

In general, an MOP with m objectives is described as