Information Sciences 262 (2014) 215–222

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Observables on quantum structures ☆

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ARTICLE INFO

Article history:
Received 5 October 2012
Received in revised form 27 August 2013
Accepted 3 September 2013
Available online 13 September 2013

Keywords:
Effect algebra
Observable
Monotone σ-completeness
Loomis–Sikorski Theorem
Effect-tribe
Riesz Decomposition Property

ABSTRACT

An observable on a quantum structure is any σ-homomorphism of quantum structures from the Borel σ-algebra into the quantum structure. We show that our partial information on an observable known only for all intervals of the form $(-\infty, t)$ is sufficient to derive the whole information about the observable defined on quantum structures like σ-MV-algebras, σ-lattice effect algebras, Boolean σ-algebras, monotone σ-complete effect algebras with the Riesz Decomposition Property, the effect algebra of effect operators of a Hilbert space, and systems of functions – effect-tribes.

1. Introduction

Quantum mechanics is the most effective tool for description of the physical world. Recently new discoveries have been found for its applications in information and computation science [23]. As it was mentioned in the special issue of Information Sciences [21] dedicated to quantum structures, quantum information science is a new field of science and technology, combining and drawing on the disciplines of physical science, mathematics, computer science, quantum computing, and engineering. Quantum information science aims to understand how certain fundamental principles of physics discovered early in the 20th century can be harnessed to dramatically improve the acquisition, transmission, and processing of information. One of the roots of quantum information science comes from the quantum structures modeling, see [12].

In a classical physical system, the observable events form a Boolean algebra. However, the quantum mechanical event structure for quantum mechanics is no more a Boolean algebra. Therefore, Birkhoff and von Neumann [3] introduced orthomodular lattices as the event structure describing quantum mechanical experiments. Later, orthomodular lattices and orthomodular posets were considered as the standard quantum logic [31]. In the nineties of the last century, two equivalent quantum structures, D-posets, [8], and effect algebras, [14], were introduced. They are generalizations of many structures which arise in quantum mechanics, in particular, of orthomodular lattices and MV-algebras. The quantum structures are also frequently used in information sciences, see e.g. [6,32,33].

☆ The paper has been supported by Center of Excellence SAS – Quantum Technologies-, ERDF OP R& D Project meta-QUTE ITMS 26240120022, Slovak Research and Development Agency under the Contract APVV-0178-11, the Grant VEGA No. 2/0059/12 SAV, CZ.1.07/2.3.00/20.0051, and VEGA 1/0621/11.
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0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2013.09.014

Effect algebras were introduced by Foulis and Bennett [14] to model quantum mechanical events and they are a basic notion of theory of quantum structures. An effect algebra is an algebraic structure endowed with the primary notion - addition of mutually excluding events. For example, if E is a Boolean algebra or an algebra of subsets, $a + b$ means that a and b are disjoint and $a + b := a \vee b$. A prototypical example of effect algebras is the set $\mathcal{E}(H)$ of all Hermitian operators of a Hilbert space H which lie between the zero and the identity operators. During the last two decades effect algebras became the most important part of theory of quantum structures which studies orthomodular lattices, orthomodular posets, Boolean algebras, MV-algebras, etc. under the same roof. We recall that MV-algebras were introduced by Chang [5] as the equivalent algebraic semantics for Łukasiewicz infinite-valued calculus.

To gain information about a measurement, in particular about a quantum measurement, we need an analogue of a random variable, which is in our case an observable, a kind of a σ-homomorphism of effect algebras from the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ into the given quantum structure preserving only partial addition and limits of monotone sequences. In quantum mechanics, an observable is simply a POV-measure (=positive operator valued measure). Observables were studied by many authors with different aims. In the last period in the series of papers [24,17,18], the authoresses concentrated to observables studied on lattice effect algebras and σ-MV-algebras exhibiting spectral properties and smearing of fuzzy observables by sharp observables using a kind of a Markov kernel. In [8] there were presented conditions for a functional calculus of observables in D-posets.

As we have stressed, an observable is a basic notion of theory of quantum structures, and now it is studied also in different kinds of algebraic structures. For example, observables in the realm of MV-algebras are exhibited in [7,20,27,26].

In the present paper, we concentrate to the question whether our information about an observable x corresponding to a quantum measurement known only for all intervals of the form $(-\infty, t)$, $t \in \mathbb{R}$, is sufficient to derive the whole information about x. We show that this is possible and we prove it for observables on σ-MV-algebras, σ-lattice effect algebras, Boolean σ-algebras, quantum logics, monotone σ-complete effect algebras with the Riesz Decomposition Property (RDP), $\mathcal{E}(H)$, and effect-tribes. This was possible to show thanks to some generalizations of the famous Loomis–Sikorski Theorem from Boolean σ-algebras to σ-MV-algebras, [9,22], and to monotone σ-complete effect algebras with RDP, [10]. We recall that RDP roughly speaking means that every two decompositions of the unity have a common refinement, and thus RDP is a kind of a weak form of the distributivity.

The paper is organized as follows. In Section 2, we gather elements of theory of effect algebras, and the body of the paper is in Section 3. Section 4 indicates some possible use of observables.

2. Elements of effect algebras and MV-algebras

We recall that according to [14], an *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation + and with two constant elements 0 and 1 such that, for all a, b, $c \in E$,

(i) $a + b$ is defined in E if and only if $b + a$ is defined, and in such a case $a + b = b + a$;
(ii) $a + b$ and $(a + b) + c$ are defined if and only if $b + c$ and $a + (b + c)$ are defined, and in such a case $(a + b) + c = a + (b + c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;
(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \leqslant b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\leqslant$ is a partial ordering on E, and we write $c := b - a$. It is clear that $a' = 1 - a$ for any $a \in E$. As a primary source of information about effect algebras we can recommend the monograph [12]. It is important to note that an effect algebra is not necessarily a lattice. We recall that a *homomorphism* from an effect algebra E_1 into another one E_2 is any mapping $h: E_1 \to E_2$ such that (i) $h(1) = 1$ (ii) if $a + b$ is defined in E_1 so is defined $h(a) + h(b)$ in E_2 and $h(a + b) = h(a) + h(b)$.

For example, let E be a system of fuzzy sets on Ω, that is $E \subseteq [0,1]^{\Omega}$, such that (i) $1 \in E$, (ii) $f \in E$ implies $1 - f \in E$, and (iii) if $f, g \in E$ and $f(\omega) \leqslant 1 - g(\omega)$ for any $\omega \in \Omega$, then $f + g \in E$. Then E is an effect algebra of fuzzy sets which is not necessarily a Boolean algebra. In addition, if G is an Abelian partially ordered group written additively, $u \in G^+$, then $\Gamma(G,u) := [0,u] = \{g \in G: 0 \leqslant g \leqslant u\}$ is an effect algebra with $0 = 0$, $1 = u$ and + is the group addition of elements if it exists in $\Gamma(G,u)$. In particular, if $G = \mathbb{R}$, the group of real numbers, then $[0,1] = \Gamma(\mathbb{R},1)$ is the standard effect algebra of the real interval $[0,1]$; it is an interval effect algebra.

We say that an effect algebra E satisfies the Riesz Decomposition Property (RDP for short) if for all $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are four elements c_{11}, c_{12}, c_{21}, c_{22} such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

We define $\sum_{i=1}^{n} a_i := a_1 + \cdots + a_n$, if the element on the right-hand exists in E. A system of elements $\{a_i: i \in I\}$ is said to be *summable* if, for any finite set F of I, the element $a_F := \sum_{i \in F} a_i$ is defined in E. If there is an element $a := \sup\{a_F: F$ is a finite subset of $I\}$, we call it the *sum* of $\{a_i: i \in I\}$ and we write $a = \sum\{a_i : i \in I\}$.

An effect algebra E is *monotone σ-complete* if, for any sequence $a_1 \leqslant a_2 \leqslant \cdots$, the element $a = \bigvee_n a_n$ is defined in E (we write $\{a_n\} \nearrow a$). Equivalently, every summable sequence has a sum.

If E and F are two monotone σ-complete effect algebras, a homomorphism $h: E \to F$ is said to be a *σ-homomorphism* if $\{a_n\} \nearrow a$ implies $\{h(a_n)\} \nearrow h(a)$, for $a, a_1, \ldots, \in E$.