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# A class of implications related to Yager's $f$-implications

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ABSTRACT

In the paper we introduce $I_{U,f,g}$ implications generated by two fuzzy negations and a uninorm. These implications are generalizations of the formula $\neg(A \wedge \neg B)$ from the classical logic and further we generalize the $f$-generated fuzzy implications introduced by Yager. We study basic properties of the newly proposed implications as well as properties of their $\varphi$-transformations.



## 1. Introduction

A fuzzy implication is a mapping $I$: $[0,1]^2 \rightarrow [0,1]$ that generalizes the classical implication to fuzzy logic, in a similar way as t-norms (t-conorms) generalize the classical conjunction (disjunction). Of course there is also a straightforward application of fuzzy implications in other branches of 'fuzzy mathematics', e.g., in fuzzy reasoning, in constructions of fuzzy controllers, and, if we assume more general $L$-fuzzy implications, in studying of fuzzy topological structures [6].

There exist many ways how to construct fuzzy implications (see e.g., [2–5,12,14,18,22,24]). It is well known that there is a one-to-one correspondence between fuzzy implications and disjunctions such that for an arbitrary fuzzy implication $I$ there exists a disjunction $D$ for which

$$I(1-x,y) = D(x,y) \quad \text{and vice versa} \quad D(1-x,y) = I(x,y).$$

Generalizing this idea we get a representation of various classes of fuzzy implications by a quadruple $(*,f,g,h)$ in the form $I(x,y) = h(f(x) * g(y))$, where $f$, $g$, $h$ are appropriate one variable functions and $*$ is a suitable binary operation, e.g., a t-norm, a uninorm, ordinary arithmetic operations +, $\cdot$, etc. For example, for a quadruple $(S,N,\mathrm{id},\mathrm{id})$, where $S$ is a t-conorm and $N$ is a negation, we get an (S,N)-implication, for $(+,-f,f,f^{(-1)})$, where $f$ is an additive generator of a generated t-norm we get an R-implication, for a quadruple $(\cdot,\mathrm{id},f,f^{-1})$, where $\cdot$ is the ordinary multiplication and $f$ is a generator of an Archimedean t-norm, we get Yager's implications $I_{T_P,f^{-1},f}$. In the paper we study implications generated by a quadruple $(U,\mathrm{id},f,g)$, where $U$ is a conjunctive uninorm and $f$, $g$ are negations. These implications generalize in a straightforward way Yager's $f$-implications with $f(0) < \infty$.

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We focus here on their basic properties with respect to chosen functions $f$, $g$ and a uninorm $U$ as well as properties of their $\varphi$-transformations.

The paper is organized as follows. In Section 2 we review basic definitions and known results used in the rest of our paper. In Section 3 we propose the mentioned new class of implications $(U, \mathrm{id}, f, g)$. Moreover, we provide here a partial characterization of its properties with respect to negations $f$, $g$ and a conjunctive uninorm $U$.

## 2. Preliminaries

In literature, we can find several definitions of fuzzy implications. In this paper we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [8]. For more details the reader can consult [1] or [14].

**Definition 2.1.** A function $I$: $[0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

(I1) $I$ is decreasing in its first variable,
(I2) $I$ is increasing in its second variable,
(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

Fuzzy implications are closely related to other fuzzy connectives, e.g., fuzzy negations, fuzzy conjunctions and disjunctions. Note that fuzzy conjunctions and disjunctions are usually modelled by means of triangular norms and triangular conorms, respectively.

**Definition 2.2** (*see, e.g.,* [8]). A decreasing function $N$: $[0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$, $N(1) = 0$. A fuzzy negation $N$ is called

1. strict if it is strictly decreasing and continuous in $[0,1]$,
2. strong if it is an involution, i.e., if $N(N(x)) = x$ for all $x \in [0,1]$.

**Definition 2.3** (*see, e.g.,* [21]). A triangular norm $T$ (t-norm for short) is a commutative, associative, monotone binary operator on the unit interval $[0,1]$, fulfilling the boundary condition $T(x,1) = x$, for all $x \in [0,1]$.

**Remark 2.4.** Note that, for an involutive (strong) negation $N$, the $N$-dual operation to a t-norm $T$ defined by $S(x,y) = N(T(N(x),N(y)))$ is called t-conorm. For more information, see, e.g., [13].

Uninorms, as a generalization of triangular norms and conorms, were introduced by Yager and Rybalov in [25].

**Definition 2.5.** An associative, commutative and increasing operation $U$: $[0,1]^2 \to [0,1]$ is called a uninorm, if there exists $e \in [0, 1]$, called the neutral element of $U$, such that

$$U(x,e) = x \quad \text{for all } x \in [0,1].$$

If $U(1,0) = 0$ holds, $U$ is called a conjunctive uninorm. If $U(1,0) = 1$ holds, $U$ is called a disjunctive uninorm. Conjunctive and disjunctive uninorms are dual to each other. For an arbitrary disjunctive uninorm $U$ and a strict negation $N$ its $N$-dual conjunctive uninorm is given by

$$U_N^d(x,y) = N^{-1}(U(N(x),N(y))). \tag{1}$$

There were several different classes of uninorms introduced in the literature, namely representable uninorms, pseudo-continuous uninorms, idempotent uninorms, uninorms continuous in $]0,1[^2$, etc. (see, e.g., [9,11,20]).

**Definition 2.6.** Let $e \in\, ]0,1[$. A uninorm $U$ is said to be representable with neutral element $e$ if there exists a strictly increasing bijection $h: [0, 1] \to \mathbb{R} \cup \{-\infty, \infty\}$ such that $h(e) = 0$ and

$$U(x,y) = h^{-1}(h(x) + h(y)).$$

A representable uninorm $U$ is said to be conjunctive if we set $\infty - \infty = -\infty$.
A representable uninorm $U$ is said to be disjunctive if we set $\infty - \infty = \infty$.

The bijection $h$ is said to be an additive generator of the uninorm $U$.

**Remark 2.7.** Let $U$ be a representable uninorm. Then we get immediately (see, e.g., [9]) that its additive generator is given uniquely up to a positive multiplicative constant.