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A note on the mutual independence of the properties in the characterization of residual fuzzy implications derived from left-continuous uninorms

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ABSTRACT

The axiomatic characterization of residual implications derived from left-continuous uninorms was presented by Aguiló, Suñer and Torrens. They have also investigated the mutual independence of the properties in this characterization. In this work we will give two examples which solve this problem.

1. Introduction

Implication functions are probably the most important operations in fuzzy logic. They are useful in fuzzy control, approximate reasoning, decision support systems, data mining and many other fields. The mutual independence of the properties in the characterization of residual implications is an open problem. However, the same problem for the case of t-norms (mutual independence of the properties in the characterization of R-implications) has been recently solved (see [5,2, Table 2.7]). There are some of these properties for the case of uninorms that are independent to each other (see [1]). Our aim is to show missing examples which finally solve this problem.

First we recall some definitions.

Definition 1 (*see [7]*). A function $U : [0,1]^2 \to [0,1]$ is called a uninorm if U is associative, commutative, increasing in each variable and there exists some element $e \in [0,1]$, called neutral element, such that

$$U(e,x) = x, \qquad x \in [0,1].$$

Definition 2 (*see [2]*). A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies:

$$\text{if } x_1 \leqslant x_2, \text{ then } I(x_1,y) \geqslant I(x_2,y), \quad x_1,x_2,y \in [0,1], \tag{I1}$$
$$\text{if } y_1 \leqslant y_2, \text{ then } I(x,y_1) \leqslant I(x,y_2), \quad x,y_1,y_2 \in [0,1], \tag{I2}$$
$$I(0,0) = I(1,1) = 1 \text{ and } I(1,0) = 0. \tag{I3}$$

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Definition 3 (*see [2,6]*). A function $I:[0,1]^2\to[0,1]$ is called an RU-operation if there exists a uninorm U such that

$$I(x,y)=\sup\{t\in[0,1]\mid U(x,t)\leqslant y\},\qquad x,y\in[0,1].$$

Proposition 1 (see [3], Proposition 7). *Let U be a uninorm and I_U its RU-operation. Then I_U is a fuzzy implication (called RU-implication) if and only if the following condition holds:*

$$U(x,0)=0,\qquad x\in[0,1).$$

There are some properties connected with RU-implications (see [1, Proposition 4]):

1. Exchange principle, i.e.,

$$I(x,I(y,z))=I(y,I(x,z)),\qquad x,y,z\in[0,1]. \tag{EP}$$

2. The ordering property for the neutral element e, i.e.,

$$e\leqslant I(x,y)\iff x\leqslant y,\qquad x,y\in[0,1]. \tag{OP$_e$}$$

One of the most important results connected with RU-implications is their characterization.

Theorem 1 (cf. [1, Theorem 4], see also [4, Theorem 1.14] and [2, Theorem 2.5.17]). *Let $I:[0,1]^2\to[0,1]$ be a function and $e\in(0,1]$. Then the following statements are equivalent:*

(i) *I is an RU-implication derived from a left-continuous uninorm U with a neutral element e.*
(ii) *I satisfies (I1), (OP$_e$), (EP) and $I(x,\cdot)$ is right-continuous for all $x\in[0,1]$.*
Moreover, in this case the uninorm U must be conjunctive and it is given by:

$$U(x,y)=\inf\{z\in[0,1]:I(x,z)\leqslant y\},\qquad x,y\in[0,1].$$

The mutual-independence of the properties in the theorem above has been indicated as an open problem in [1], where some examples of independence between these properties were shown. For the case of t-norms ($e=1$) the problem has been recently solved (see [5,2, Table 2.7]).

Problem 1 [1, after Remark 2], [2, Problem 2.7.2]. Prove or disprove by giving a counter example:

Let $I:[0,1]^2\to[0,1]$ be any function that satisfies both (EP) and (OP$_e$). Then the following statements are equivalent:

(i) I satisfies (I2).
(ii) I is right-continuous in the second variable.

Table 1
The mutual independence of some properties in Theorem 1.

Function F	(I1)	(EP)	(OP$_e$)	Right-continuity
$F(x,y)=\begin{cases}1, & \text{if } x=0\\ y, & \text{if } 0<x\leqslant y \text{ and } e<y\leqslant 1\\ e, & \text{if } 0<x\leqslant y\leqslant e\\ e-x+y, & \text{if } 0<y<x\leqslant e\\ 0, & \text{if } 0=y<x\leqslant e \text{ or } (e<x\leqslant 1 \text{ and } 0\leqslant y<x)\end{cases}$	✓	✓	✓	×
$F(x,y)=\begin{cases}1, & \text{if } 0\leqslant x\leqslant y<e \text{ or } (0\leqslant x\leqslant ey \text{ and } e\leqslant y\leqslant 1)\\ y, & \text{if } (y<x\leqslant 1 \text{ and } 0\leqslant y<e) \text{ or } (ey<x\leqslant y \text{ and } e\leqslant y\leqslant 1)\\ ey, & \text{if } e\leqslant y<x\leqslant 1\end{cases}$	×	✓	✓	✓
$F(x,y)=\begin{cases}1, & \text{if } 0\leqslant x\leqslant y<1\\ 0, & \text{if } 0\leqslant y<x\leqslant 1\end{cases}$	✓	×	✓	✓
$F(x,y)=\max(1-x,y),\ x,y\in[0,1]$	✓	✓	×	✓