Information Sciences 213 (2012) 1–13

Contents lists available at SciVerse ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

A measurement theory view on the granularity of partitions

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ARTICLE INFO

Article history:
Received 15 December 2009
Received in revised form 25 March 2012
Accepted 25 May 2012
Available online 7 June 2012

Keywords:
Expected granularity
Granular computing
Granularity
Hartley entropy
Information theory
Partition lattice

ABSTRACT

Measurement of granularity is one of the foundational issues in granular computing. This paper investigates a class of measures of granularity of partitions. The granularity of a set is defined by a strictly monotonic increasing transformation of the cardinality of the set. The granularity of a partition is defined as the expected granularity of all blocks of the partition with respect to the probability distribution defined by the partition. Many existing measures of granularity are instances of the proposed class. New measures of granularity of partitions are also introduced.

1. Introduction

Granular computing concerns problem solving and information processing at multiple levels of granularity [1,39,55,59,60,62]. A partition is perhaps the simplest granulation scheme and hence has attracted attentions from many researchers [52,58]. A partition of a universal set consists of a family of nonempty and pair-wise disjoint sets whose union is the universe. Each block in a partition may be viewed as a granule by ignoring the differences between elements in the block. Two examples of partition based granular computing theories are rough set theory [33,34,37] and quotient space theory of problem solving [63].

A crucial issue of granular computing is the search for an appropriate level of granularity by ignoring unimportant and irrelevant details. This requires the measurement of granularity. In a partition based model, it is the measurement of granularity of a partition. From a measurement-theoretic point of view [43], a measure of granularity of partitions is a quantification of some intuitive, qualitative relationships between partitions. Studies on measuring granularity of partitions mainly focus on two relations. A partition can be equivalently defined by an equivalence relation (i.e., a reflexive, symmetric, and transitive relation). Based on the set–inclusion relationship between the corresponding equivalence relations, one can define a partial order, called the refinement–coarsening relation, or the specialization–generalization relation, on the set of all partitions [34,58]. Wierman [50] introduced an equivalence relation on the set of all partitions of a set based on the notion of size–isomorphic. If two partitions have the same number of blocks and there exists a bijection that connects blocks of the same sizes, the two partitions are said to be size–isomorphic. It seems reasonable to require that any measure of granularity of partitions must reflect the refinement–coarsening relationship [7,30,31,50,56], namely, a refined partition has a lower granularity. Sometimes, the measure also needs to preserve the size–isomorphic relationship [23,25,50,66], namely, all equivalent partitions have the same level of granularity.

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0020-0255/$ - see front matter
http://dx.doi.org/10.1016/j.ins.2012.05.021

From a measurement-theoretic perspective, this paper makes new contributions to the study of the granularity of partitions. We have three specific objectives that have not been considered in existing studies. First, we will establish a measurement-theoretic basis for a study of granularity. Intuitive understanding and properties of *finer than* relations are introduced on subsets of a universal set and on partitions of the universe, respectively. The results from the measurement theory enable us to establish the conditions (i.e., axioms) for the measurements of the *finer than* relations. Second, we introduce a new class of measures of granularity of partitions based on the expected granularity of blocks in a partition. The new measure reveals the inherent relationships between the granularity of blocks in a partition and the granularity of the partition. Third, we show that many existing measures are instances of the proposed class.

The rest of the paper is organized as follows. Section 2 reviews results from measurement theory to establish a theoretical basis for a study of granularity. Ordinal measurements of granularity are considered. Section 3 examines, in general, measures of granularity of a set and measures of granularity of a partition. Section 4 proposes a new class of measures of granularity of partitions, namely, granularity of a partition is the expected granularity of its blocks. Section 5 provides a critical review of existing studies on measures of granularity of partitions, introduces new measures, and comments on applications.

2. Measurement-theoretic foundations

To lay down a solid foundation for the measurement of granularity of partitions, we draw results from measurement theory. Pertinent concepts are reviewed based on books by Krantz et al. [17], Roberts [43], Fishburn [11], and French [12]. The ordinal measurements of granularity are examined.

2.1. An overview

When measuring an attribute of a class of objects or events, we may associate numbers with the individual objects so that the properties of the attribute are faithfully represented as numerical properties [17]. The properties are usually described by certain qualitative relations and operations. Formally, measurement may be studied based on homomorphisms between an empirical system and a numerical system.

A *relational system* (*structure*) is a set together with one or more relations (operations) on that set [43]. That is, a relational system is an ordered $(p+q+1)$-tuple $\mathcal{A} = (O, R_1, \ldots, R_p, \circ_1, \ldots, \circ_q)$, where O is a set, $R_1, \ldots, R_p$ are (not necessarily binary) relations on O, and $\circ_1, \ldots, \circ_q$ are binary operations on O. We call a relational system a numerical relational system if O is the set (or a subset) of real numbers. Operations can be considered as a special kind of relations. For convenience, we separate them from other relations. For modeling measurement, we start with an observed or empirical system $\mathcal{A}$ and seek a mapping into a numerical relational system $\mathcal{B}$ that preserves or faithfully reflects all the properties of the relations and operations in $\mathcal{A}$. Consider two relational systems, an empirical (a qualitative) system $\mathcal{A} = (O, R_1, \ldots, R_p, \circ_1, \ldots, \circ_q)$, and a numerical system $\mathcal{B} = (V, R'_1, \ldots, R'_p, \circ'_1, \ldots, \circ'_q)$. A function $f : O \to V$ is called a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if, for all $a_1, \ldots, a_{r_i} \in \mathcal{A}$,

$$R_i(a_1, \ldots, a_{r_i}) \Longleftrightarrow R'_i(f(a_1), \ldots, f(a_{r_i})), \quad i = 1, \ldots, p,$$

and for all $a, b \in \mathcal{A}$,

$$f(a \circ_j b) = f(a) \circ'_j f(b), \quad j = 1, \ldots, q.$$

That is, through a homomorphism f, the properties of the empirical system are truthfully reflected in the numerical system. Thus, f provides a measurement of the empirical system.

Consider a simple empirical system $(O, \prec, \circ)$, where O is a set of objects, $\prec$ is an ordering relation on O and $\circ$ is a binary relation on O. The numerical relation system is $(\Re, <, +)$, where $\Re$ is the set of real numbers, $<$ is the usual "smaller than" relation and $+$ is the arithmetic operation of addition. A numerical assignment $\phi(\cdot)$ is a homomorphism which maps O into $\Re$, $\prec$ into $<$, and $\circ$ into $+$ in such a way that $<$ preserves the properties of $\prec$, and $+$ preserves the properties of $\circ$ as stated by the conditions:

$$a \prec b \Longleftrightarrow \phi(a) < \phi(b),$$
$$\phi(a \circ b) = \phi(a) + \phi(b).$$

Measurement of length or weight of objects may be considered as an example using the empirical system $(O, \prec, \circ)$. For measuring length, we have a set of straight, rigid rods. The relation $\prec$ is interpreted as a "shorter than" relation. For two rods a and $b, a \prec b$ if we place them side by side with one of the endpoints coinciding and find that a does not extend to reach, or beyond, b on the opposite endpoint; operation $\circ$ denotes the operation of "concatenation" of two rods, that is, $a \circ b$ is obtained by joining a and b end-to-end along a straight line. For measuring weight, $\prec$ denotes the relation "lighter than" and $\circ$ denotes the operation of the "combination" of two objects. In addition to the basic ordering and operation, one needs to consider more properties for measuring length or weight [43].

There are three fundamental issues in measurement theory [12,17,43]. Suppose we are seeking a quantitative representation of an empirical system. The first step, naturally, is to define the relations and operations to be represented. We must describe the valid use of these relations and operations. The consistency properties to be preserved are known as *axioms*. The set of axioms characterizing the empirical system should be complete in the sense that every consistency property that we