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Solutions to the functional equation $I(x,y) = I(x, I(x,y))$ for a continuous D-operation ☆

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ABSTRACT

We present a study, in a fuzzy logic framework, of the fuzzy implication operations involved in the iterative Boolean-like law $I(x,I(x,y)) = I(x,y)$. And then, a full characterization is given of this functional equation based on a Dishkant operation (or D-operation for short), where the D-operation is generated by a continuous t-norm T, a continuous t-conorm S, and a strong negation N. Together with the work by Shi et al. (Y. Shi, D. Ruan, E.E. Kerre, On the characterization of fuzzy implications satisfying $I(x,y) = I(x,I(x,y))$, Information Sciences, 177 (2007) 2954–2970), this work provides a complete characterization of the equation for the usual implication operations currently known.

1. Introduction

It is well-known that a fuzzy set is a generalization of a classical set, but not all Boolean identities are always valid in any fuzzy set theory [12,13]. For example, although $A \bigcap \neg A = \emptyset$ holds in a Boolean algebra, $T(a, \neg a)$, in the standard fuzzy set theory $([0,1], T, S, N)$, does not always hold [1]. Accordingly, Trillas et al. [12] analyzed some non-standard aspects in the construction of fuzzy set theories and dealt with the derived Boolean properties of fuzzy operations, among which, although the iterative Boolean-like laws have induced Boolean properties, they are not always true in any standard fuzzy set theory.

Alsian and Trillas [1] formulated iterative Boolean-like laws in fuzzy logic with some variables appearing several times. These laws are derived from Boolean identities to which no simplifications have been made, such as the application of idempotency or distributivity, absorption, etc. Furthermore, considering that fuzzy implication operations play an important role in fuzzy logic, Shi et al. [10] discussed some iterative Boolean-like laws with fuzzy implication operations, and then transformed the derived Boolean law $p \to (p \to q) = p \to q$ into the functional equation

$$I(x, I(x,y)) = I(x,y). \tag{1}$$

Finally, they characterized the solutions of this equation with respect to the S- and R-implication and QL-operation. Note that, if $y = 0$ and $I(x,y) = S(N(x),y)$, Eq. (1) becomes $I(x, N(x)) = N(x)$, which is one of the necessary conditions of the inclusion grade indicator to construct fuzzy entropy and has been characterized thoroughly in [4]. Hence it is important to investigate Eq. (1).

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On the other hand, like the QL-operations, the D-operations originate from orthomodular lattices. Although D-operations are the contrapositive symmetry of QL-operations with respect to a strong negation [8,13], neither operation itself has contrapositive symmetry (see Remarks 4, 8, 9). Hence, Trillas introduced the D-operation and D-implication in fuzzy set theory and pointed out that they can be widely applied in the fields of fuzzy logic and fuzzy reasoning [13]. Since then, they have been considered widely and studied systemically [8,12]. These are our motivations for characterizing all solutions of Eq. (1) based on D-operations.

Some basic definitions and theorems are given in the second section of this paper, while in the third section, we characterize the sufficient and necessary conditions of a D-operation to satisfy Eq. (1), where the D-operation is generated by a continuous t-norm T, a continuous t-conorm S, and a strong negation N. We also find that a certain method to study $I_{S,T,N}$ in [10] does not apply for $I^{S,T,N}$ (see Remark 9).

2. Preliminaries

To ensure that this paper is self-explanatory, we briefly recall some of the concepts and results utilized in the rest of this paper.

Definition 1 [4]. A continuous mapping $\varphi : [0,1] \to [0,1]$ is said to be an *order automorphism* on the unit interval [0,1], if

(1) φ is strictly increasing, and
(2) $\varphi(0) = 0$ and $\varphi(1) = 1$.

Definition 2 [2]. Two mappings $F, G : [0,1]^n \to [0,1]$, where $n \in \mathcal{N}$, are said to be *conjugate*, if there exists an order automorphism φ on the unit interval [0,1] such that $G = F_\varphi$, where $F_\varphi(x_1, x_2, \ldots, x_n) = \varphi^{-1}(F(\varphi(x_1), \varphi(x_2) \ldots, \varphi(x_n)))$, $x_1, x_2, \ldots, x_n \in [0,1]$.

Definition 3 [5]. A continuous function $N : [0,1] \to [0,1]$ is called a *strong negation*, if it is strictly decreasing, involutive and satisfies $N(0) = 1$ and $N(1) = 0$. We refer to the strong negation $N(x) = 1 - x$ as the standard strong negation and denote it by N_0.

Remark 1. For any strong negation N, there exists a unique equilibrium point $e \in (0,1)$ such that $N(e) = e$.

Theorem 1 [4]. *A fuzzy negation N is strong if and only if there exists an order automorphism φ on the unit interval [0,1] such that N and N_0 are conjugate. In other words, $N(x) = \varphi^{-1}(1 - \varphi(x))$ holds for all $x \in [0,1]$.*

Definition 4 [6,7]. A binary function $T : [0,1]^2 \to [0,1]$ is called a *triangular norm* (t-norm for short) if, for all $x, y, z \in [0,1]$, the following four axioms are satisfied:

(1) $T(x,y) = T(y,x)$ (commutativity),
(2) $T(T(x,y),z) = T(x,T(y,z))$ (associativity),
(3) $T(x,y) \leqslant T(x,z)$, whenever $y \leqslant z$ (monotonicity),
(4) $T(x,1) = x$ (boundary condition).

Example 1 [6]. The four basic t-norms are T_M, T_P, T_L, and T_D given by, respectively:

$$T_M(x,y) = \min(x,y); \quad T_L(x,y) = \max(x+y-1, 0);$$
$$T_P(x,y) = x * y; \quad T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise}. \end{cases}$$

Definition 5 [10]. A t-norm T is called a *continuous Archimedean t-norm* if it is continuous and satisfies $T(x,x) < x$ for all $x \in (0,1)$.

A continuous t-norm T is said to be *strict* if it holds that $T(x,y) < T(x,z)$ for all $x > 0$ and $y < z$. While a t-norm T is said to be *nilpotent* if it is continuous and for all $x \in (0,1)$ there exists some $n \in \mathcal{N}$ such that $x_T^n = 0$, where x_T^n is defined recursively as $x_T^1 = x, x_T^n = T(x, x_T^{n-1})$ for all $n \geqslant 2$.

It is well-known that every continuous Archimedean t-norm is either strict or nilpotent. Now, let us recall in the following, the full characterizations of strict, nilpotent and continuous t-norms, respectively.

Theorem 2 [6].

(i) *A function $T : [0,1]^2 \to [0,1]$ is a strict t-norm if and only if there exists an order automorphism φ on the unit interval [0,1] such that T is conjugate with the product t-norm T_P, i.e., it holds that $T(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$ for all $x, y \in [0,1]$.*