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Biconic aggregation functions

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ABSTRACT

We introduce a new method to construct aggregation functions. These aggregation functions are called biconic aggregation functions with a given diagonal (resp. opposite diagonal) section and their construction is based on linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points (0,1) and (1,0) (resp. (0,0) and (1,1)). Subclasses of biconic aggregation functions such as biconic semi-copulas, biconic quasi-copulas and biconic copulas are studied in detail.

1. Introduction

Aggregation functions have become very popular over the last years due to their wide range of applications in several areas of research. Their main role appears in applied sciences, such as image processing, decision making, control theory, information retrieval, etc. [3,15,24]. In general, aggregation functions are used to convert finitely many input values into a single representative output value. To increase modelling flexibility, new methods to construct aggregation functions are being proposed continuously in the literature [2,6,8,22,24–26].

A (binary) aggregation function A is an increasing $[0,1]^2 \to [0,1]$ function that preserves the bounds, i.e. $A(0,0) = 0$ and $A(1,1) = 1$. Special classes of aggregation functions are of particular interest, such as semi-copulas [20,21], triangular norms [1,32], quasi-copulas [23,27,33] and copulas [1,34]. They are all conjunctors, in the sense that they extend the classical Boolean conjunction. Recall that an aggregation function A is a semi-copula if it has 1 as neutral element, i.e. $A(x,1) = A(1,x) = x$ for any $x \in [0,1]$. Evidently, any semi-copula S has 0 as annihilator, i.e. $S(0,x) = S(x,0) = 0$ for any $x \in [0,1]$. A semi-copula S is a triangular norm (t-norm for short) if it is commutative and associative. The aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ and $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are examples of t-norms. Moreover, for any semi-copula S the inequality $T_{\mathbf{D}} \leqslant S \leqslant T_{\mathbf{M}}$ holds. A semi-copula S is a quasi-copula if it is 1-Lipschitz continuous, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \leqslant x'$ and $y \leqslant y'$, it holds that

$$|S(x',y') - S(x,y)| \leqslant |x' - x| + |y' - y|.$$

A semi-copula S is a copula if it is 2-increasing, i.e. for any $x, x', y, y' \in [0,1]$ such that $x \leqslant x'$ and $y \leqslant y'$, it holds that

$$V_S([x,x'] \times [y,y']) := S(x',y') + S(x,y) - S(x',y) - S(x,y') \geqslant 0.$$

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V_S is called the volume of the rectangle $[x,x'] \times [y,y']$. Any copula is a quasi-copula since the 2-increasingness of a semi-copula implies its 1-Lipschitz continuity. The (quasi-)copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ (the Łukasiewicz t-norm) with $T_{\mathbf{L}}(x,y) = \max(x + y - 1,0)$, are respectively the greatest and the smallest (quasi-)copulas, i.e. for any (quasi-)copula C, it holds that $T_{\mathbf{L}} \leqslant C \leqslant T_{\mathbf{M}}$.

The surface of the aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is constituted from their zero-set and linear segments connecting the upper boundary curve of their zero-set to the point (1,1,1). In earlier work, this observation has led to the notion of conic aggregation functions [30]. Characteristic for the aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is also that their surface is constituted from linear segments connecting their diagonal section to the points (0,1,0) and (1,0,0). Similarly, their surface is constituted from linear segments connecting their opposite diagonal section to the points (0,0,0) and (1,1,1). Inspired by these observations, we introduce a new method to construct aggregation functions. These aggregation functions are constructed by linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points (0,1) and (1,0) (resp. (0,0) and (1,1)).

This paper is organized as follows. In the following section we recall some definitions and properties of diagonal and opposite diagonal sections. In Section 3 we introduce the definition of a biconic function with a given diagonal section and characterize the class of biconic aggregation functions. In Sections 4–7, we characterize the classes of biconic semi-copulas, biconic quasi-copulas, biconic copulas and singular biconic copulas. For biconic copulas, we provide simple expressions for Spearman's ρ, Kendall's τ and Gini's γ in Section 8. In Section 9, we study the aggregation of biconic (semi-, quasi-) copulas. The class of biconic functions with a given opposite diagonal section is introduced in Section 10. Finally, some conclusions are given.

2. Diagonal and opposite diagonal sections

The diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\delta_F: [0,1] \to [0,1]$ defined by $\delta_F(x) = F(x,x)$. In order to characterize the diagonal section of (quasi-)copulas, the following class of functions was considered. A diagonal function [17,18] is a function $\delta: [0,1] \to [0,1]$ satisfying the following properties:

(**D1**) $\delta(0) = 0$, $\delta(1) = 1$;
(**D2**) δ is increasing;
(**D3**) for all $x \in [0,1]$, it holds that $\delta(x) \leqslant x$;
(**D4**) δ is 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$|\delta(x') - \delta(x)| \leqslant 2|x' - x|.$$

The set of all diagonal functions is denoted by $\mathcal{D}$. In the following propositions, we describe the diagonal section of aggregation functions, semi-copulas, quasi-copulas and copulas.

Proposition 1 [35]. *A function $\delta: [0,1] \to [0,1]$ is the diagonal section of a (quasi-)copula if and only if it satisfies properties* **D1**–**D4**, *i.e.* $\delta \in \mathcal{D}$.

The copula C_δ, introduced in Proposition 1, is the greatest symmetric copula with diagonal section δ [18,19,35].
The set of all $[0,1] \to [0,1]$ functions that satisfy properties **D1**–**D3** is denoted by $\mathcal{D}_S$.

Proposition 2 [21]. *A function $\delta: [0,1] \to [0,1]$ is the diagonal section of a semi-copula if and only if it satisfies properties* **D1**–**D3**, *i.e.* $\delta \in \mathcal{D}_S$.

Note that for a function $\delta \in \mathcal{D}_S$, the function C_δ defined in Proposition 1 has neutral element 1 if and only if $\delta(x) \geqslant 2x - 1$ for any $x \in [1/2,1]$. In fact, the last inequality holds for the class of diagonal sections of quasi-copulas and copulas. Therefore, for a given $\delta \in \mathcal{D}_S$, the function C_δ defined in Proposition 1 need not be a semi-copula in general.
The set of all $[0,1] \to [0,1]$ functions that satisfy properties **D1** and **D2** is denoted by $\mathcal{D}_A$.

Proposition 3. *A function $\delta: [0,1] \to [0,1]$ is the diagonal section of an aggregation function if and only if it satisfies properties* **D1** *and* **D2**, *i.e.* $\delta \in \mathcal{D}_A$.

Proof. Let A be an aggregation function with diagonal section δ. Clearly, δ satisfies properties **D1** and **D2**. Conversely, consider a function $\delta: [0,1] \to [0,1]$ that satisfies properties **D1** and **D2** and consider the function $A_\delta: [0,1]^2 \to [0,1]$ defined by

$$A_\delta(x,y) = \frac{\delta(x) + \delta(y)}{2}.$$

One easily verifies that A_δ is an aggregation function with diagonal section δ. □

Similarly, the opposite diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\omega_F: [0,1] \to [0,1]$ defined by $\omega_F(x) = F(x, 1 - x)$. In order to characterize the opposite diagonal section of (quasi-)copulas, the following class of functions was considered. An opposite diagonal function [9] is a function $\omega: [0,1] \to [0,1]$ satisfying the following properties: