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Long paths in hypercubes with a quadratic number of faults

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ABSTRACT

A path between distinct vertices u and v of the n-dimensional hypercube Q_n avoiding a given set of f faulty vertices is called long if its length is at least $2^n - 2f - 2$. We present a function $\phi(n) = \Theta(n^2)$ such that if $f \leqslant \phi(n)$ then there is a long fault-free path between every pair of distinct vertices of the largest fault-free block of Q_n. Moreover, the bound provided by $\phi(n)$ is asymptotically optimal. Furthermore, we show that assuming $f \leqslant \phi(n)$, the existence of a long fault-free path between an arbitrary pair of vertices may be verified in polynomial time with respect to n and, if the path exists, its construction performed in linear time with respect to its length.

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1. Introduction

The n-dimensional hypercube Q_n is a graph with all binary strings of length n as vertices, an edge joining two vertices that differ in a single position. The applications of hypercubes as prospective interconnection networks for parallel or distributed computing [9] inspired the investigation of their fault-tolerant properties. This paper studies long paths in hypercubes that avoid a given set of faulty vertices, corresponding to the nodes of the network that are unavailable.

It is well-known that for any pair u, v of distinct vertices of Q_n there is a path between them of length at least $2^n - 2$ [5,10]. Suppose we are given a set of f faulty vertices of Q_n. What can be said about the maximum length of a path between u and v avoiding all faults? We call such a path *long* if its length is at least $2^n - 2f - 2$. The motivation for this concept lies in the observation that if all faulty vertices are of the same parity as u or v, then no fault-free path between u and v can be longer than the long path between them.

The existence of long fault-free paths in faulty hypercubes was studied by Fu [4] who showed that such a path exists between an arbitrary pair of distinct fault-free vertices provided $f \leqslant n - 2$. Kueng et al. [8] improved that bound to $f \leqslant 2n - 5$ under the assumption that $n \geqslant 3$ and every fault-free vertex has at least two fault-free neighbors. The research of linear bounds on f has been satisfactory completed by a recent result of Fink and Gregor [3]. They proved that if $f \leqslant 2n - 4$ and $n \geqslant 5$, a long fault-free path between u and v exists if and only if both u and v possess at least one fault-free neighbor different from u and v. Moreover, the latter bound on f is tight, as for every $n \geqslant 5$ there is a configuration of $2n - 3$ faulty vertices and two distinct fault-free vertices u, v satisfying the above necessary condition, but no fault-free path between them is long. The second result of [3] says that if $n \leqslant 15$ and the number of faulty vertices f is at most $\phi(n) = \frac{n^2}{10} + \frac{n}{2} + 1$, then there exists a fault-free cycle of length at least $2^n - 2f$. Our aim is to continue on this result.

In this paper we show that if $f \leqslant \phi(n)$ then there is a long fault-free path between every pair of distinct vertices of the largest fault-free block of Q_n. As a corollary, we obtain the above quoted result of [3] on the existence of a long fault-free

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cycle (Section 5). Moreover, the bound provided by $\phi(n)$ is asymptotically optimal in the sense that for every $n \geqslant 6$, there is a configuration of $\Theta(n^2)$ faulty vertices such that no long fault-free path between any pair of vertices exists (Section 6). Turning to the computational complexity, we show that if $f \leqslant \phi(n)$ then the existence of a long fault-free path between an arbitrary pair of vertices may be verified in polynomial time with respect to n and, if the path exists, then its construction performed in linear time with respect to its length (Section 7). The paper is concluded with possible applications of this problem and directions for further research.

2. Preliminaries

The graph-theoretic terms used in this paper but not defined below can be found e.g. in [6]. Throughout the paper, n always is a positive integer while $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

Given a graph G with vertices $V(G)$ and a set $V \subseteq V(G)$, $G - V$ denotes the induced subgraph of G on the set $V(G) \setminus V$. Maximal connected subgraphs of G are *components* of G. A *cutvertex* of G is a vertex v such that $G - \{v\}$ has more components than G. A connected graph on at least three vertices which contains no cutvertices is called *biconnected. Set* $B \subseteq V(G)$ is called a *block* of G if either B is vertex set of a maximal biconnected subgraph of G, or B consists of endvertices of an edge which is not contained in any biconnected subgraph. A block is called *largest* if it has maximum number of vertices among all blocks of G. Given a set $\mathscr{F}$ of faulty vertices of G, the blocks of $G - \mathscr{F}$ are called *fault-free blocks* of G.

A one-to-one sequence $a = x_0, x_1, \ldots, x_n = b$ of vertices such that x_i and x_{i+1} are adjacent for all $i = 0, 1, \ldots, n-1$ is a *path between* a and b of *length n*. We denote such a path, its vertices and its length by $P_{ab}, V(P_{ab})$ and $l(P_{ab})$, respectively. Let P_{ab} and P_{bc} be paths such that $V(P_{ab}) \cap V(P_{bc}) = \{b\}$. Then $P_{ab} + P_{bc}$ denotes the path between a and c, obtained as a concatenation of P_{ab} with P_{bc} (where b is taken only once). Observe that the operation + is associative. Path R is called a *subpath* of a path P if $P = S + R + T$ for some paths S, T. If P_{ad} and P_{bc} are paths such that $V(P_{ad}) \cap V(P_{bc}) = \{b, c\}$ and $P_{ad} = P_{ab} + b, c + P_{cd}$, then $P_{ad}(P_{bc})$ denotes the path $P_{ab} + P_{bc} + P_{cd}$.

In this paper we deal with the n-dimensional hypercube Q_n, which is a graph with all binary strings of length n as vertices, an edge joining two vertices that differ in a single position. The *parity* $p(v)$ of a vertex $v = v_1, v_2 \ldots v_n$ is defined as $|\{i | v_i = 1\}| \bmod 2$. Note that the distance $d(u, v)$ of vertices u and v of Q_n is odd if and only if $p(u) \neq p(v)$.

Given a string $v = v_1, v_2 \ldots v_n$ and a set $D \subseteq [n]$, we use v_D to denote the string $v_{j_1}, v_{j_2} \ldots v_{j_d}$ where $j_1, j_2, \ldots, j_d$ is an increasing sequence of all elements of D. Given a set $D \subseteq [n]$ of size d and a vertex $u \in V(Q_{n-d})$, the subgraph of Q_n induced by vertex set $\{v \in V(Q_n) | v_{\overline{D}} = u\}$ where $\overline{D} = [n] \setminus D$ is denoted by $Q_D(u)$ and called a *subcube of dimension d*. Subcubes $Q_D(u)$ and $Q_D(v)$ are called *adjacent* if u and v are adjacent vertices of Q_{n-d}. Note that

- $Q_D(u)$ is isomorphic to Q_d,
- If $Q_D(u)$ and $Q_D(v)$ are adjacent subcubes, then an arbitrary vertex in one of them has a unique neighbor in the other.

Given a set $\mathscr{F} \subseteq V(Q_n)$ of faulty vertices, $\mathscr{F}_D(u)$ denotes the set $\mathscr{F} \cap V(Q_D(u))$. If there is a unique largest fault-free block of $Q_D(u)$, we denote it by $B_D(u)$.

Recall that a fault-free path P_{ab} in Q_n with f faulty vertices is called *long* if

$$l(P_{ab}) \geqslant 2^n - 2f - 2.$$

Since Q_n is a bipartite graph, we have $l(P_{ab}) \equiv d(a, b) \bmod 2$. In particular, if $d(a, b)$ is odd and P_{ab} is long then

$$l(P_{ab}) \geqslant 2^n - 2f - 1. \tag{2.1}$$

For the purpose of algorithms description we employ two fundamental data structures:

- queue Q; item x is added to the tail of the queue using Enqueue(x, Q) and retrieved from the head using Dequeue(Q) in a constant time,
- trie T over the alphabet $\{0,1\}$; binary string x of length n is inserted into T using Insert(x, T) in $O(n)$ time, we assume that each leaf of T may include additional information related to the string x it represents (e.g. the number of occurrences of x).

3. Lemmas

This section is devoted to auxiliary results that will be used later to verify certain steps of our algorithm. For the basis of our construction we employ the following special case of a more general result by Fink and Gregor.

Theorem 3.1 [3]. *Let $\mathscr{F}$ be a set of at most six faulty vertices in Q_5. Then for every pair of distinct fault-free vertices u and v, there is a long fault-free path between them if and only if both u and v possess at least one fault-free neighbor different from u and v.*

For larger dimensions we use the strategy of splitting the faulty hypercube into subcubes with only a small number of faults. As observed in [3], this task may be performed easily using a recent result by Wiener [13, Theorem 2.5]. As it was originally stated for set systems, here we provide an equivalent formulation using the above introduced terminology.

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