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Topological properties of generalized approximation spaces ☆

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ABSTRACT

Rough set theory is a powerful mathematical tool for dealing with inexact, uncertain or vague information. The core concepts of rough set theory are information systems and approximation operators of approximation spaces. Approximation operators draw close links between rough set theory and topology. This paper concerns generalized approximation spaces via topological methods and studies topological properties of rough sets. Classical separation axioms, compactness and connectedness for topological spaces are extended to generalized approximation spaces. Relationships among separation axioms for generalized approximation spaces and relationships between topological spaces and their induced generalized approximation spaces are investigated. An example is given to illustrate a new approach to recover missing values for incomplete information systems by regularity of generalized approximation spaces.

1. Introduction

Rough set theory is first proposed by Pawlak [14,15] for dealing with vagueness and granularity in information systems. It has been successfully applied to various fields such as pattern recognition, machine learning, data mining and automated knowledge acquisition. The core concepts of rough set theory are approximations. In terms of the lower and upper approximations in the theory, knowledge hidden in information systems may be discovered and expressed in the form of decision rules. So far, the theory has drawn attentions of researchers and practitioners in various fields of science and technology.

In classical rough set theory, a pair (U,R) is called an approximation space, where U is a finite non-empty set called universe and R an equivalent relation on U. But the requirement that R is an equivalent relation is too restrictive to be satisfied in many situations and which limits the application of rough set theory. So, various generalizations of Pawlak's rough sets have been made by replacing equivalent relations with kinds of binary relations and many results about generalized rough sets with universes being finite were obtained [16–18,24,27,29]. We call a pair (U,R) a generalized approximation space (GA-space for short), where U is a non-empty set (maybe infinite) and R is a binary relation on U. Early studies on GA-spaces with infinite universes can be found in [6,13,20].

Topology [3], one of the most important subjects in mathematics, provides mathematical tools and interesting topics in studying information systems and rough sets [2,6,7,10,19–23,28]. Many authors studied relationships between (fuzzy) topologies and the structures of rough sets based on (fuzzy) relations [2,6,7,19,20,23,26]. It is known that the pair of lower and upper approximation operators induced by a reflexive and transitive relation (namely, a preorder) is exactly the pair of interior and closure operators of a topology [2,7,20]. So, given a GA-space (U,R) with R being a preorder, one gets an induced

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topological space $(U,\mathcal{T}_R)$ and then we call the GA-space (U,R) a topological GA-space. Chen and Zhang in [2] investigated topological properties of Pawlak approximation spaces (U,R) with U being finite and R an equivalent relation. But for general GA-spaces, relatively few results about their concrete topological properties were obtained. In this paper, we consider topological properties of GA-spaces in general cases that U may be infinite and R any binary relation on U. We will first examine the classical separation axioms, compactness and connectedness of topological GA-spaces and then extend them to GA-spaces.

Topological properties of GA-spaces may have some applications in information science. In practical, it often happens that some attribute values for an object of an information system are missing. For such incomplete information systems, Kryszkiewicz [8,9] and Salama [21] studied the rule generation and information recovery by rough set approach and topological method, respectively. We in this paper give an example to illustrate a new approach to recover missing attribute values for incomplete information systems by regularity of GA-spaces.

The rest of this paper is organized as follows. Section 2 presents basic concepts and facts of GA-spaces and topological spaces. Section 3 characterizes separation axioms T_i $(i=0,1,2)$ of topological GA-spaces and extends them into general GA-spaces. In Section 4, we investigate the regularity and normality for GA-spaces and their applications in information science. Section 5 is concerned with the compactness and connectedness for GA-spaces. And finally, some concluding remarks appear in Section 6.

2. Preliminaries on GA-spaces and topological spaces

Let (U,R) be a GA-space. The relation $R^c=\{(x,y)\in U\times U|(x,y)\notin R\}$ denotes the complement relation of R. For $x\in U$, the sets $R_s(x)=\{y\in U|xRy\}$ and $R_p(x)=\{y\in U|yRx\}$ are respectively called the successor neighborhood and predecessor neighborhood of x [25]. If xRy and xRz imply yRz for all $x,y,z\in U$, then R is called a Euclidean relation. If R is reflective and transitive, then R is called a preorder. If a preorder R is also antisymmetric, i.e., xRy and yRx imply $x=y$ for all $x,y\in U$, then R is called a partial order. For a preorder R on U and $A\subseteq U$, we set $\downarrow A=\{y\in U|yRx$ for some $x\in A\}$ and $\uparrow A=\{y\in U|xRy$ for some $x\in A\}$. For a singleton $\{x\}$, we use $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. We say that A is a lower set if $A=\downarrow A$ and that A is an upper set if $A=\uparrow A$.

Let $(X,\mathcal{T})$ be a topological space. We use A° to denote the interior of a subset $A\subseteq X$, A^- to denote the closure of A. If $A\in\mathcal{T}$ and $x\in A$, then A is called an open neighborhood of point x. If $B\in\mathcal{T}$ and $A\subseteq B$, then we call B an open neighborhood of A.

Given a topological space $(X,\mathcal{T})$, the specialization order [4] $R_{\mathcal{T}}$ on X is defined by: for all $x,y\in U$, $xR_{\mathcal{T}}y$ iff $x\in\{y\}^-$ iff $y\in W$ for any open neighborhood W of x. It is easy to see that $R_{\mathcal{T}}$ is a preorder. For a topological space, if we concern an order without special statements, we always refer to the specialization order.

A topological space is called an Alexandrov space [5] iff its topology is closed under arbitrary intersections. It is known that in an Alexandrov space X, the open sets are just the upper sets and the closed sets are just the lower sets. So, in an Alexandrov space X, every subset $A\subseteq X$ has a smallest open neighborhood $\uparrow A$.

Lower and upper approximations are key notions in GA-spaces. We recall definitions and basic properties of lower and upper approximations operators of GA-spaces.

Definition 2.1 [24]. Let (U,R) be a GA-space. For $A\subseteq U$, the lower and upper approximations of A in (U,R) are respectively defined as

$$\underline{R}A=\{x\in U|R_s(x)\subseteq A\},\quad \overline{R}A=\{x\in U|R_s(x)\cap A\neq\emptyset\}.$$

The operators $\underline{R},\overline{R}:\mathcal{P}(U)\to\mathcal{P}(U)$ are respectively called the lower and upper approximation operators in (U,R), where $\mathcal{P}(U)$ is the power set of U.

Lemma 2.2 ([11–13]). *Let (U,R) be a GA-space. Then the lower and upper approximation operators $\underline{R}$ and $\overline{R}$ have the following properties.*

(1) $\underline{R}(\sim A)=\sim(\overline{R}A)$, $\overline{R}(\sim A)=\sim(\underline{R}A)$, *where* $\sim A$ *is the complement of* $A\subseteq U$.
(2) $\underline{R}(U)=U$, $\overline{R}(\emptyset)=\emptyset$.
(3) *Let* $\{A_i|i\in I\}\subseteq\mathcal{P}(U)$. *Then* $\underline{R}(\bigcap_{i\in I}A_i)=\bigcap_{i\in I}\underline{R}A_i$, $\overline{R}(\bigcup_{i\in I}A_i)=\bigcup_{i\in I}\overline{R}A_i$.
(4) *If* $A\subseteq B\subseteq U$, *then* $\underline{R}A\subseteq\underline{R}B$, $\overline{R}A\subseteq\overline{R}B$.

Proofs of the following two lemmas are similar to those for the finite cases and are omitted.

Lemma 2.3. *Let (U,R) be a GA-space. Then the following statements are equivalent:*

(1) *R is reflexive;*
(2) $\underline{R}A\subseteq A$ *for all* $A\subseteq U$;
(3) $A\subseteq\overline{R}A$ *for all* $A\subseteq U$.