Information Sciences 181 (2011) 1411–1425

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

On convergence of the multi-objective particle swarm optimizers

Prithwish Chakraborty [a], Swagatam Das [a,*], Gourab Ghosh Roy [a], Ajith Abraham [b]

[a] Dept. of Electronics and Telecommunication Eng., Jadavpur University, Kolkata, India
[b] Machine Intelligence Research Labs (MIR Labs), Scientific Network for Innovation and Research Excellence, P.O. Box 2259, Auburn, Washington 98071-2259, USA

A R T I C L E   I N F O

Article history:
Received 31 March 2010
Received in revised form 25 October 2010
Accepted 27 November 2010
Available online 27 December 2010

Keywords:
Evolutionary algorithms
Particle swarm optimization
Pareto optimality
Pareto dominance
Convergence
Mathematical analysis
Z-transform

A B S T R A C T

Several variants of the particle swarm optimization (PSO) algorithm have been proposed in recent past to tackle the multi-objective optimization (MO) problems based on the concept of Pareto optimality. Although a plethora of significant research articles have so far been published on analysis of the stability and convergence properties of PSO as a single-objective optimizer, till date, to the best of our knowledge, no such analysis exists for the multi-objective PSO (MOPSO) algorithms. This paper presents a first, simple analysis of the general Pareto-based MOPSO and finds conditions on its most important control parameters (the inertia factor and acceleration coefficients) that govern the convergence behavior of the algorithm to the optimal Pareto front in the objective function space. Computer simulations over benchmark MO problems have also been provided to substantiate the theoretical derivations.

1. Introduction

The concept of particle swarm, although initially introduced for simulating social behavior commonly observed in animal kingdom, has become very popular these days as an efficient means for intelligent search and optimization. Since its advent in 1995, the particle swarm optimization (PSO) [21,22] algorithm has attracted the attention of a lot of researchers all over the world resulting into a huge number of variants of the basic algorithm as well as many parameter selection/control strategies, comprehensive surveys of which can be found in [14,10,2–4]. In a D-dimensional search space, the position vector of the ith particle is given by $X_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,D})$ and velocity of the ith particle is given by $V_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,D})$. Positions and velocities are adjusted and the objective function to be optimized, i.e. $f(X_i)$ is evaluated with the new positional coordinates at each time-step. Expressions for velocity and position of the ith individual at tth iteration in a geographical best PSO may be given as:

$$V_i(t+1) = \omega \cdot V_i(t) + \varphi_1 \cdot R_1 \cdot \left(P_i^l - X_i(t)\right) + \varphi_2 \cdot R_2 \cdot (P^g - X_i(t)) \quad (1a)$$

$$X_i(t+1) = X_i(t) + V_i(t+1) \quad (1b)$$

where P_i^l is the personal best position found so far by an individual particle and P^g represents the best position found so far by the entire swarm, for *gbest* PSO model. R_1 and R_2 are random positive numbers uniformly distributed in (0, 1) and are drawn a new for each dimension of each particle. Constants φ_1 and φ_2 are called acceleration coefficients and they determine the relative influences of the cognitive and social parts on the velocity of the particle. The particle's velocity may be optionally

* Corresponding author.
E-mail addresses: prithwish1611@gmail.com (P. Chakraborty), swagatamdas19@yahoo.co.in (S. Das), myself_gourab@yahoo.co.in (G.G. Roy), ajith.abraham@ieee.org (A. Abraham).

0020-0255/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.ins.2010.11.036

clamped to a maximum value $V_{max} = [v_{max,1}, v_{max,2}, \ldots, v_{max,D}]^T$. If in dth dimension, $|v_{i,d}|$ exceeds $v_{max,d}$ specified by the user, then the velocity of that dimension is assigned to $sign(v_{i,d}) * v_{max,d}$, where $sign(x)$ is the triple-valued signum function [10].

The first stability analysis of the particle dynamics was due to Clerc and Kennedy [6]. Van den Bergh undertook an independent theoretical analysis of the particle swarm dynamics in his Ph.D. thesis [38], published in the same year. In [6], Clerc and Kennedy considered a deterministic approximation of the swarm dynamics by treating the random coefficients as constants, and studied stable and limit cyclic behavior of the dynamics for the settings of appropriate values to its parameters. A more generalized stability analysis of particle dynamics based on Lyapunov stability theorems was undertaken by Kadirkamanathan et al. [20]. Recently Poli [28] analyzed the characteristics of a PSO sampling distribution and explained how it changes over any number of generations, in the presence of stochasticity, during stagnation. Some other significant works towards the theoretical understanding of PSO can be found in [33,17,35,5,37]. However, to the best of our knowledge, all the theoretical research works undertaken so far, are centered on the single-objective PSO algorithm, although, during the past few years, several efficient multi-objective variants of PSO have been proposed.

The field of multi-objective optimization (MO) deals with the simultaneous optimization of multiple and conflicting objective functions, without combining them in a weighted sum. The MO problems tend to be characterized by a family of alternatives, which must be considered equivalent in the absence of information concerning the relevance of each objective relative to the others. The family of solutions of an MO problem is composed of the parameter vectors, which cannot be improved in any objective without causing degradation in at least one of the other objectives, and this set is said to be the *Pareto optimal set* and its image in the objective function space is usually called the *Pareto front (PF)*. In case of several MO problems, knowledge about this set helps the decision maker in choosing the best compromise solution [7].

Recently, several MOPSO algorithms have been developed based on the Pareto optimality concept. One fundamental issue is the selection of the cognitive and social leaders (P_i and P_g) such that they can provide an effective guidance to reach the most promising Pareto front region but at the same time maintain the population diversity. For the selection procedure researchers have suggested two typical approaches: selection based on quantitative standards and random selection. In the first case, the leader is determined by some procedure, without any randomness involved, such as the Pareto ranking scheme [29], the sigma method [25] or the dominated tree [15]. However, in the random approach, the selection for a candidate is stochastic and proportional to certain weights assigned to maintain the population diversity (crowding radius, crowding factor, niche count, etc.). For example, Ray and Liew [30] choose the particles that perform better to be the leaders and the remaining particles tend to move towards a randomly selected leader from this leader group where the leader with fewer followers has the highest probability of being selected.

Coello Coello and Lechuga [8] incorporated the concept of Pareto dominance into PSO. In this case, the non-dominated solutions are stored in a secondary population and the primary population uses a randomly selected neighborhood best from this secondary population to update their velocities. The authors proposed an adaptive grid to generate well-distributed PFs and mutation operators to enhance the exploratory capabilities of the swarm [9]. Keeping the same two goals (obtaining a set of non-dominated solutions as close as possible to the PF and maintaining a well-distributed solution set along the PF), Li [24] proposed sorting the entire population into various non-domination levels such that the individuals from better fronts can be selected. In this way, the selection process pushes towards the true PF. Other authors have developed different approaches such as combining canonical PSO with auto fitness sharing concepts [32], dynamic neighborhood PSO [18], PSO with time-varying parameters for MO problems [36], PSO with preference order ranking [39], vector evaluated PSO [27], two local bests (*lbest*) based MOPSO [41] and PSO with strength Pareto approach [13]. Some intensive experimental studies on MOPSO can be found in the recent works of Durillo et al. [11,12].

In this paper we have presented a simple theoretical analysis of the general continuous multi-objective PSO algorithm. Conditions for the convergence of MOPSO to some solutions (at least one) in the Pareto optimal set have been deduced based on the non-dominated selection scheme for updating the personal best and the global best positions. The analysis provides suitable ranges of the control parameters like ω, φ_1 and φ_2 that ensures the convergence of MOPSO. Limited experimental results on 10 standard MO benchmarks on the basis of metrics like hypervolume difference [23], R-indicator [23], and IGD [40] have been provided to support the analytical results derived in the article.

2. Analytical treatment

For MOPSO, suppose n particles are randomly scattered in the search space and following Eq. (1). Expectedly, decisions like the updating of local best or global best are determined using the concept of Pareto-optimality. We assume a Pareto-based approach to be taken for implementing the selection of the globally best particle of the swarm in every iteration. The algorithm is expected to identify the set of non-dominated solutions of the population at each iteration and store the best non-dominated solutions found throughout the search process in an external archive (e.g. see the MOPSO described in [9]). The global best particle P_g may be chosen from this archive. The use of global attraction mechanisms combined with a historical archive of previously found non-dominated vectors can motivate convergence toward globally non-dominated solutions.

We attempt to investigate the convergence characteristics of the MOPSO algorithm by examining the evolution of the probability distribution of the population, based on which the search algorithm is run. Our method is inspired by the analysis reported in [16] for multi-objective evolutionary algorithms. Since in a generic PSO each dimension of a particle is perturbed independently of the other dimensions, we can say that it will not be a loss of generality if we conduct our analysis for