Information Sciences 179 (2009) 742–757

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Extended triangular norms

Janusz T. Starczewski

Department of Computer Engineering, Czestochowa University of Technology, Al. Armii Krajowej 36, 42-200 Czestochowa, Poland

ARTICLE INFO

Article history:
Received 2 May 2007
Received in revised form 31 October 2008
Accepted 5 November 2008

Keywords:
Extended triangular norms
Extension principle
Type-2 fuzzy sets
Fuzzy truth values
Fuzzy truth intervals
Fuzzy truth numbers

ABSTRACT

The paper is devoted to classical t-norms extended to operations on fuzzy quantities in accordance with the generalized Zadeh extension principle. Such extended t-norms are used for calculating intersection of type-2 fuzzy sets. Analytical expressions for membership functions of some extended t-norms are derived assuming special classes of fuzzy quantities, i.e., fuzzy truth intervals or fuzzy truth numbers. The possibility of applying these results in the construction of type-2 adaptive network fuzzy inference systems is illustrated on several examples.

1. Introduction

Triangular norms have been used widely as basic operations on fuzzy sets allowing to construct various structures of fuzzy logic systems. Within last 10 years, they have been extended to form the algebra of type-2 fuzzy sets [19].

Introduced in [22], type-2 fuzzy sets are fuzzy sets equipped with ordinary fuzzy subsets of $[0,1]$ as membership grades, henceforth called fuzzy truth values. Following this, classical triangular norms (t-norms for short) had to be fuzzified to model the intersection of type-2 fuzzy sets. The extension principle is commonly used in order to fuzzify any mathematical operation, in particular, a triangular norm [14,22]. However, such extended operation is considerably difficult to perform, as quite often there is an infinite number of combinations of input variables producing only one discrete value of the continuous range. The standard approach necessitates discretization of input domains. As a result, a discrete set of function values is obtained instead of an analytical formula.

Despite of an increasing interest in the pragmatic aspect of type-2 fuzzy sets, the lack of exact formulae for the extended t-norm hampered the development of non-interval type-2 fuzzy systems. Up to now, almost all attempts to develop fuzzy logic systems which operate on fuzzy sets of type-2 have been based on interval fuzzy sets [10] (see e.g. [1,4,11,15,20,21]), even though several years have passed since the theoretical foundation of general type-2 fuzzy logic systems was published [6]. In [10], Mendel noticed that approaches to compute operations for general type-2 fuzzy sets (with various, non-interval secondary membership functions) look very promising. Operations on triangular [16,17] and Gaussian type-2 fuzzy sets [18] have been just the first steps we made in this direction. Nevertheless, in case of other more general type-2 fuzzy logic systems, it is necessary to derive analytical formulae for extensions of t-norms, which may enrich the Generalized Theory of Uncertainty in the future [23]. This problem is studied and solved in this paper.

Hardly any combinations of t-norms and membership functions enable us to obtain exact analytical formulae for extended t-norms. Hence, the paper discusses in detail particular combinations of:

E-mail address: jasio@kik.pcz.czest.pl

0020-0255/$ - see front matter
doi:10.1016/j.ins.2008.11.009

Table 1
Analytical expressions for extended t-norms: known and original in this paper.

Extended t-norm T	Based on T_*	Arguments F and G	Reference
Arbitrary	Any	Interval value	[22]
Minimum	Minimum, product	FT number	[2,5]
Minimum	Arbitrary	FT interval	Theorem 9
Continuous	Minimum	USc FT interval	Theorem 13
Strict	Minimum	Continuous FT number	Corollary 15
Continuous	Drastic product	FT number	Theorem 23
Łukasiewicz	Continuous Archimedean	$\phi \circ f$ concave on slopes	Theorem 16
Łukasiewicz	Strict	$= \phi^{-1}(a\kappa(\frac{u-m}{a}))$	Theorem 18
Łukasiewicz	Nilpotent	$= \phi^{-1}(a\kappa(\frac{u-m}{a}))$	Theorem 20
Product	Product	Trapezoidal	Theorem 21

FT stands for fuzzy truth, USc stands for upper semicontinuous. Symbols: ϕ – additive generator of the t-norm, f – membership function, κ – monotone kernel function, a, m – constants in $[0,1]$.

- fuzzified t-norms (T in (1)),
- t-norms on which the exteded operations are based on (T_* in (1)),
- membership functions (f and g in (1)).

The extended minimum based on an arbitrary t-norm for convex and normal fuzzy sets merely exists in the literature [2,5]. The aim of this paper is to derive the exact formulae for membership functions of extended t-norms, assuming various combinations of fuzzified t-norms, extension basis, and argument forms, which convincedly enriches realizations of type-2 fuzzy logic systems, especially their network structures. The original results presented in this paper, against a background of existing formulae, are summarized in Table 1. In the table, the two initial rows refer to the well known exact formulae for extended t-norms. The following rows present our results: the extended minimum based on an arbitrary t-norm (Theorem 9 in Section 3), extended continuous t-norms based on the minimum (Theorem 13 and Corollary 15 in Section 4), the extended Łukasiewicz t-norm based on continuous Archimedean t-norms (Theorems 16, 17, 18, 19, 20 in Section 5), the extended algebraic product based on the product (Theorem 21 in Section 6), and extensions of continuous t-norms based on the drastic product (Theorem 23 in Section 7).

While constructing type-2 fuzzy systems, it is considered advantageous when an extended t-norm preserves its shapes, i.e., when it does not pull out the result outside of the class of arguments. Consequently, classes of input and output type-2 fuzzy sets can be usually represented by function parameters stored in fuzzy logic systems. For example, the knowledge about the analytical formulae for extended t-norms allows us to express firing fuzzy grades by the mean value and standard deviation. This approach reduces the computational cost of operations performed on parameters and provides opportunity for converting the type-2 fuzzy logic systems into their network structures, hence called type-2 adaptive network fuzzy inference systems.

Accordingly, in this study, the shape preserving property of analytical formulae for extended t-norms assuming special classes of arguments, i.e., fuzzy truth intervals or fuzzy truth numbers, is examined. To the best knowledge of the author, the original problems solved in this paper have not been studied before.

2. Type-2 fuzzy sets and extended triangular norms

To begin with, we formalize a type-2 fuzzy set as a generalization of a fuzzy set whose membership grades are fuzzy subsets of the unit interval.

Definition 1. A fuzzy set of type-2, $\widetilde{A}$, in the real line $\mathbb{R}$, is a vague collection of elements characterized by membership function $\mu_{\widetilde{A}} : \mathbb{R} \to \mathscr{F}([0,1])$, where $\mathscr{F}([0,1])$ is a set of all classical fuzzy sets in the unit interval $[0,1]$.

That is, each $x \in \mathbb{R}$ is associated with a secondary membership function $f_x \in \mathscr{F}([0,1])$ being a mapping $f_x : [0,1] \to [0,1]$. The fuzzy membership grade $\mu_{\widetilde{A}}(x)$ applied is called a fuzzy truth value, since its domain is the truth interval $[0,1]$. Among all possible fuzzy truth values, only fuzzy truth intervals and their particular forms, fuzzy truth numbers, express the uncertainty of memberships adequately, since the secondary membership function can be regarded either as fuzzy interval or as uncertain single membership grade. Moreover, in Section 3, we will show that for fuzzy truth intervals, extended t-norms satisfy all axioms of type-2 t-norms, which is not the case for general fuzzy truth values.

Definition 2. A fuzzy truth value F with a membership function f is called a fuzzy truth interval if it is normal, $\exists u \in [0,1]\ f(u) = 1$, and convex, $\forall u_1, u_2, \lambda \in [0,1], f(\lambda u_1 + (1-\lambda)u_2) \geqslant \min(f(u_1), f(u_2))$.

Definition 3. A fuzzy truth interval is a fuzzy truth number when the normality is additionally satisfied by a unique number, i.e., $\exists! u \in [0,1]\ f(u) = 1$.