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Numerical solution of hybrid fuzzy differential equation IVPs by a characterization theorem

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ABSTRACT

In this paper, we study hybrid fuzzy differential equation initial value problems (IVPs). We consider the problem of finding their numerical solutions by using a recent characterization theorem of Bede for fuzzy differential equations. We prove a corollary to Bede's characterization theorem and give a characterization theorem for hybrid fuzzy differential equation IVPs. Then we prove that any suitable numerical method for ODEs can be applied piecewise to numerically solve hybrid fuzzy differential equation IVPs. Numerical examples are provided which connect the new results with previous findings.

1. Introduction

The differential calculus of fuzzy-valued functions was examined by Dubois and Prade [7] and Puri and Ralescu [15]. Subsequently, fuzzy differential equations were considered by many papers including [8,18,5]. Numerical techniques were developed in [11,1–3] and others. In [4], Bede proved a characterization theorem which states that under certain conditions a fuzzy differential equation IVP is equivalent to a system of ordinary differential equations. Bede also remarked that this characterization theorem can help to numerically solve fuzzy differential equation IVPs by converting them to systems of ODEs which can then be solved by any suitable numerical method for ODEs. More specifically, in [4], Bede wrote, "in order to obtain numerical solutions of fuzzy differential equations under Hukuhara differentiability, it is not necessary to rewrite the whole literature on numerical solutions of ODEs in the fuzzy setting, but instead we can use any numerical method for the ODEs directly".

Also receiving much attention in the recent literature are hybrid systems. Hybrid systems evolve in continuous time like differential systems but undergo fundamental changes in their governing equations at a sequence of discrete times. When the continuous time dynamics of a hybrid system is given by fuzzy differential equations the system is called a hybrid fuzzy differential system. For analytical results on hybrid fuzzy differential equations, see [9,17,10]. Pederson and Sambandham [12,13] study the Euler and Runge–Kutta numerical methods, respectively for hybrid fuzzy differential equations. In some sense, Pederson and Sambandham [12,13] "rewrite the whole literature on numerical solutions of ODEs" in the hybrid fuzzy setting, focusing on the Euler and Runge–Kutta methods, respectively. In contrast, the contribution of this paper is to extend Bede's characterization theorem to hybrid fuzzy differential equations and then use this result to numerically solve these

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systems *by any suitable method for ODEs*. The importance of converting a hybrid fuzzy differential equation IVP to a hybrid system of ODEs is that then any suitable numerical method for ODEs may be implemented. This paper shows that it is not necessary to restrict attention to one particular method such as the Euler method [12] or the Runge–Kutta method [13] when considering hybrid fuzzy differential equation IVPs.

This paper is organized as follows. In Section 2, we provide some background on fuzzy numbers and fuzzy differential equations. We also prove a corollary to Bede's characterization theorem. In Section 3, we review the hybrid fuzzy differential equation IVPs and give a characterization theorem for hybrid fuzzy differential equation IVPs. In Section 4, we prove that one-step explicit numerical methods for ODEs which are numerically stable can be applied piecewise to numerically solve hybrid fuzzy differential equation IVPs. In Section 5, we present numerical examples based on examples in [11,3,12,13].

2. Preliminaries

First we review some standard results about fuzzy numbers. Let E^1 denote the set of all functions $u : \mathbb{R} \to [0,1]$ such that u satisfies (i)–(iv):

(i) u is normal (there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$),
(ii) u is fuzzy convex (for $x, y \in \mathbb{R}$ and $\lambda \in [0,1], u(\lambda x + (1-\lambda)y) \geqslant \min\{u(x), u(y)\}$),
(iii) u is upper semicontinuous,
(iv) $[u]^0$, the closure of $\{x \in \mathbb{R} : u(x) > 0\}$, is compact.

For $0 < \alpha \leqslant 1$, define the α-level set $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geqslant \alpha\}$. Let $P_K(\mathbb{R})$ denote the collection of all nonempty compact, convex subsets of $\mathbb{R}$. Then the α-level sets $[u]^\alpha$ are in $P_K(\mathbb{R})$ for $0 \leqslant \alpha \leqslant 1$. Let $d_H(A,B)$ be the Hausdorff distance between sets $A, B \in P_K(\mathbb{R})$. Then

$$d(u,v) = \sup_{0 \leqslant \alpha \leqslant 1} d_H([u]^\alpha, [v]^\alpha)$$

is a metric in E^1 and (E^1, d) is a complete metric space (by results in [6,16]).

For $x, y \in E^1$ if there exists a $z \in E^1$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^1$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^1$ such that the limits (taken in the metric space (E^1, d))

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and both equal $F'(t)$. See [8,10] for details.

For $u \in E^1$ and $r \in [0,1]$, let $[u]^r = [u_-^r, u_+^r]$. Next we review one of the main results from Bede [4] (let $\|\cdot\|$ denote the usual Euclidean norm).

Theorem 2.1 [4]. *Let us consider the fuzzy initial value problem (FIVP)*

$$\begin{cases} x' = f(t,x), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $f : [t_0, t_0 + a] \times E^1 \to E^1$ is such that

(i) $[f(t,x)]^r = [f_-^r(t, x_-^r, x_+^r), f_+^r(t, x_-^r, x_+^r)]$,
(ii) f_-^r *and* f_+^r *are equicontinuous (that is, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f_\pm^r(t,x,y) - f_\pm^r(t_1,x_1,y_1)| < \epsilon$ for all $r \in [0,1]$, whenever $(t,x,y),(t_1,x_1,y_1) \in [t_0, t_0+a] \times \mathbb{R}^2$ and $\|(t,x,y) - (t_1,x_1,y_1)\| < \delta$) and uniformly bounded on any bounded set,*
(iii) *there exists an $L > 0$ such that*

$$|f_\pm^r(t,x_1,y_1) - f_\pm^r(t,x_2,y_2)| \leqslant L \max\{|x_2 - x_1|, |y_2 - y_1|\} \text{ for all } r \in [0,1].$$

Then the FIVP (2.1) and the system of ODEs

$$\begin{cases} (x_-^r(t))' = f_-^r(t, x_-^r(t), x_+^r(t)), \\ (x_+^r(t))' = f_+^r(t, x_-^r(t), x_+^r(t)), \\ x_-^r(t_0) = (x_0)_-^r, \\ x_+^r(t_0) = (x_0)_+^r \end{cases} \tag{2.2}$$

are equivalent.

Next we prove a corollary to Theorem 2.1 of [4]. The purpose of Corollary 2.2 below is not to make an essential improvement of Theorem 2.1 but rather to give alternate conditions under which the FIVP (2.1) and the system of ODEs (2.2) are equivalent.