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Uncertainty measures for interval type-2 fuzzy sets

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Abstract

Fuzziness (entropy) is a commonly used measure of uncertainty for type-1 fuzzy sets. For interval type-2 fuzzy sets (IT2 FSs), centroid, cardinality, fuzziness, variance and skewness are all measures of uncertainties. The centroid of an IT2 FS has been defined by Karnik and Mendel. In this paper, the other four concepts are defined. All definitions use a Representation Theorem for IT2 FSs. Formulas for computing the cardinality, fuzziness, variance and skewness of an IT2 FS are derived. These definitions should be useful in IT2 fuzzy logic systems design using the principles of uncertainty, and in measuring the similarity between two IT2 FSs.

Keywords: Uncertainty measure; Interval type-2 fuzzy set; Centroid; Cardinality; Fuzziness; Entropy; Variance; Skewness

1. Introduction

As pointed out by Zadeh [87], "*uncertainty is an attribute of information*". He proposed to use the generalized theory of uncertainty (GTU) to handle it. "*In GTU, uncertainty is linked to information through the concept of granular structure – a concept which plays a key role in human interaction with the real world [26,78,86]... Informally, a granule of a variable X is a clump of values of X which are drawn together by indistinguishability, equivalence, similarity, proximity or functionality. For example, an interval is a granule. So is a fuzzy interval...*"

To use fuzzy sets (FSs) as granules in GTU, there is a need to quantify the uncertainty associated with them. Klir [33] states that "*once uncertainty (and information) measures become well justified, they can very effectively be utilized for managing uncertainty and the associated information. For example, they can be utilized for extrapolating evidence, assessing the strength of relationship between given groups of variables, assessing the influence of given input variables on given output variables, measuring the loss of information when a system is simplified, and the like*".

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Three basic principles of uncertainty have been developed to guide the use of uncertainty measures in different situations [33,25]:

(1) The *principle of minimum uncertainty*, which states that solutions with the least loss of information should be selected, can be used in simplification and conflict resolution problems.
(2) The *principle of maximum uncertainty*, which states that a conclusion should maximize the relevant uncertainty within constraints given by the verified premises, is widely used within classical probability framework [14,15,56].
(3) The *principle of uncertainty invariance*, which states that the amount of uncertainty should be preserved in each transformation of uncertainty from one mathematical framework to another, is widely studied in the context of probability–possibility transformations [21,32,35,65].

However, as pointed out by Cross and Sudkamp [17], *"the quantification of the degree of uncertainty in a FS depends upon the type of uncertainty one is trying to measure and on the particular measure selected for that type of uncertainty"*. Many uncertainty measures of type-1 (T1) FSs have been proposed. Among them, *fuzziness* (entropy) [17,34] is frequently used, and it will be studied in this paper.

In addition to fuzziness, *centroid, cardinality, variance* and *skewness* are also important characteristics of T1 FSs. For example, Dubois and Prade [19] point out that *"cardinality is a natural tool for capturing the meaning of linguistic quantifiers* [80–85,77] *and to provide satisfactory answers to queries pertaining to quantification, of the form* '*How many X's are A*' , '*Are there more X's which are A than X's which are B,' etc*". These queries [74] "*occur in computing with words, communication with data bases and information/intelligent systems, modeling the meaning of imprecise quantifiers in natural language statements, decision-making in a fuzzy environment, analysis of grey-tone images, clustering, etc*". These four characteristics can also be used to measure the distance or similarity between two T1 FSs. For example, Wenstøp [64] uses the centroid and the cardinality of T1 FSs to measure their distance. This enables one FS to be found from a group of T1 FSs B_i $(i = 1, \ldots, N)$ that most resembles a target T1 FS A. Bonissone [6,7] uses a two-step approach to solve the same problem. In his first step, four measures – centroid, cardinality, fuzziness and skewness – are used to identify several FSs from the N B_i which are close to A.

Recently, there has been a growing interest in type-2 (T2) fuzzy set and system theory [79,46,47]. The membership grades of a T2 FS are T1 FSs in $[0, 1]$ instead of crisp numbers. Since the boundaries of T2 FSs are blurred, they are especially useful in circumstances where it is difficult to determine an exact membership grade [46]. To date, interval T2 (IT2) FSs are the most widely used T2 FSs, and have been used successfully for decision making [76,55,59,66], time-series forecasting [46,4], survey processing [46,3,42], document retrieval [8], speech recognition [88,45], noise cancellation [12,54], word modeling [50,72,42], clustering [57], control [71,70,38,24,13,43,20,41,58,11,44,1], wireless communication [40,60], webshopping [23], linguistic summarization of database [53,52], etc.

Though the above applications have demonstrated that IT2 FSs are better at modeling uncertainties than T1 FSs, uncertainty measures for IT2 FSs have not been extensively studied. Centroid, cardinality, fuzziness, variance and skewness are all uncertainty measures for IT2 FSs because each of them is an interval (see Section 3), and the length of the interval is an indicator of uncertainty, i.e. the larger (smaller) the interval, the more (less) the uncertainty. Once these uncertainty measures are defined for IT2 FSs, their applications in T1 FSs can be extended to IT2 FSs, e.g. the centroid and cardinality of IT2 FSs have been used in [69] to define a vector similarity measure for IT2 FSs.

The centroid of an IT2 FS has been well-defined and studied by Karnik and Mendel [28]. Because the centroid of an IT2 FS has no closed-form solution, they developed iterative algorithms, now called Karnik–Mendel (KM) Algorithms, to compute it. The cardinality of an IT2 FS was introduced in [69]. For completeness, the centroid and cardinality are again introduced in this paper. Additionally, the other three uncertainty measures of IT2 FSs – fuzziness, variance and skewness – are defined and shown how to be computed.

The rest of this paper is organized as follows: Section 2 provides background material. Section 3 gives definitions of centroid, cardinality, fuzziness, variance and skewness for IT2 FSs, and explains how to compute them. Section 4 draws conclusions. The KM Algorithms are given in the Appendix.