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On answering the question "Where do I start in order to solve a new problem involving interval type-2 fuzzy sets?"

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ABSTRACT

This paper, which is tutorial in nature, demonstrates how the *Embedded Sets Representation Theorem* (RT) for a general type-2 fuzzy set (T2 FS), when specialized to an interval (I)T2 FS, can be used as the starting point to solve many diverse problems that involve IT2 FSs. The problems considered are: set theoretic operations, centroid, uncertainty measures, similarity, inference engine computations for Mamdani IT2 fuzzy logic systems, linguistic weighted average, person membership function approach to type-2 fuzzistics, and Interval Approach to type-2 fuzzistics. Each solution obtained from the RT is a *structural solution* but is not a *practical computational solution*, however, the latter are always found from the former. It is this author's recommendation that *one should use the RT as a starting point whenever solving a new problem involving IT2 FSs* because it has had such great success in solving so many such problems in the past, and it answers the question "Where do I start in order to solve a new problem involving IT2 FSs?"

1. Introduction

Interval type-2 fuzzy sets[1,2] (IT2 FSs) are now very widely used[3] (e.g., [3,11,13,24,26,29,30,34,39–43,45,54,58,62–65]). When a researcher faces a new problem involving such fuzzy sets a natural question for her or him to ask is "Where do I start in order to solve this problem?" This paper provides an answer to this question. To begin, though, some background material is needed about type-2 and interval type-2 fuzzy sets, after which we return to this question.

A type-2 fuzzy set (T2 FS) $\tilde{A}$ can be represented in different ways [26]. The *point-valued representation* (which is usually the starting point for understanding or describing a general T2 FS) is one in which the membership function (MF) of $\tilde{A}$ is specified at every point in its 2D domain of support, i.e.

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) \,|\forall x \in X,\ \forall u \in J_x \subseteq [0,1]\} \tag{1}$$

In (1) x is the primary variable; J_x is called the *primary membership* of x—it is usually a closed interval of real numbers that are contained within $[0,1]$, but for some values of x it may only be a single value; and, u is called the *primary membership variable*

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[1] For the convenience of the reader, all abbreviations (and their meanings) that are used in this paper are collected together in Table 1.

[2] IT2 FSs are equivalent to *interval-valued fuzzy sets* (IVFS), which were introduced first by Zadeh [59], who called them "fuzzy sets with interval-valued membership functions." There are many references about IVFSs, most notably [8,1].

[3] http://www.type2fuzzylogic.org/ lists more than 500 T2 publications. Only a small sampling from them are listed here.

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(or the *secondary variable*). While useful as a starting point for obtaining the other representations, (1) does not seem to be useful for much of anything else.

The *vertical slice representation* focuses on each value of the primary variable x, and expresses (1) as:

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x)/x \tag{2}$$

$$\mu_{\tilde{A}}(x) = \int_{u \in J_x \subseteq [0,1]} f_x(u)/u \tag{3}$$

$\mu_{\tilde{A}}(x)$ is called a *secondary MF* or a *vertical slice*, and $f_x(u)$ is called the *secondary grade*. The vertical slice representation is extremely useful for computation and can also be useful for theoretical studies.

There is even a very new *alpha-plane representation* for $\tilde{A}$ [19] that has so far been used to develop a new way to perform centroid type-reduction for a general T2 FS; however, because this representation is not used in this paper, we refer the readers to [19] for its details.

Finally, there is a so-called *wavy-slice representation* (which could also be called an *embedded T2 FS representation*) [33] that is most valuable in theoretical studies because it quickly leads to the structure of the solution to a new problem, after which practical procedures are developed to compute that solution. The wavy-slice representation has also been called the *Mendel–John Representation*, and because it is the starting point for the rest of this paper, it is stated next, but for discrete universes of discourse.

Theorem 1 (*Representation Theorem [33]*). *Assume that primary variable x is sampled at N values, $x_1, x_2, \ldots, x_N$, and at each of these values its primary membership variable u_i is sampled at M_i values, $u_{i1}, u_{i2}, \ldots, u_{iM_i}$. Let $\tilde{A}_e^j$ denote the jth T2 embedded set*[4] *for T2 FS $\tilde{A}$, i.e.,*

$$\tilde{A}_e^j \equiv \left\{ \left(x_i, \left(u_i^j, f_{x_i}(u_i^j)\right)\right), \quad u_i^j \in \{u_{ik}, k = 1, \ldots, M_i\}, \quad i = 1, \ldots, N \right\} \tag{4}$$

in which $f_{x_i}(u_i^j)$ is the secondary grade at u_i^j. Note that $\tilde{A}_e^j$ can also be expressed as

$$\tilde{A}_e^j = \sum_{i=1}^{N} \left[f_{x_i}\left(u_i^j\right)/u_i^j \right] \Big/ x_i \quad u_i^j \in \{u_{ik}, k = 1, \ldots, M_i\} \tag{5}$$

Then $\tilde{A}$ can be represented as the union of its T2 embedded sets, i.e.,

$$\tilde{A} = \bigcup_{j=1}^{n_A} \tilde{A}_e^j \tag{6}$$

$$n_A = \prod_{i=1}^{N} M_i \quad \square \tag{7}$$

This representation of a T2 FS, in terms of much simpler T2 FSs, the embedded T2 FSs, is not recommended for computational purposes, because it would require the explicit enumeration of the n_A embedded T2 FSs and n_A can be astronomical.

In practice, general T2 FSs are at this time still too difficult to use, although much research is underway to rectify this, e.g., [4,5,9,10,19]. Consequently, only a special kind of T2 FS is usually used—an *interval T2 FS* (IT2 FS)—for which all of the secondary grades equal one.

Returning to the question "Where do I start in order to solve a new problem involving IT2 FSs?" we shall demonstrate that *the Representation Theorem (RT) specialized to IT2 FSs is a very good answer*.

This paper brings many (scattered) results together in one place for the first time so that the reader can see the usefulness of approaching a new theoretical problem for IT2 FSs by starting with the RT; hence, in that sense it is a tutorial. However, the idea of using the RT as a starting point to solve any new problem involving IT2 FSs is an outgrowth of its past successes, and in that sense is a new contribution.

The rest of this paper is organized as follows: Section 2 provides the RT for IT2 FSs; Section 3, which is the main section of the paper, explains how the RT in Theorem 2 can obtain the structure of solutions for many problems involving IT2 FSs; examples are given in Section 4 and Section 5 draws conclusions.

2. Representation of an IT2 FS

An IT2 FS $\tilde{A}$ is completely described [26,29] by its lower and upper MFs, $\underline{\mu}_{\tilde{A}}(x)$ and $\bar{\mu}_{\tilde{A}}(x)$, respectively. The *footprint of uncertainty* (FOU) of an IT2 FS is described in terms of these MFs, as

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{x \in X} \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x) \right] \tag{8}$$

[4] An embedded T2 FS is a T2 FS that has only one primary membership at each x_i. It is also called a wavy-slice [33].