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Kernel shapes of fuzzy sets in fuzzy systems for function approximation ☆

Qiang Luo *, Wenqiang Yang, Dongyun Yi

Department of Mathematics, National University of Defense Technology, Changsha, Hunan 410073, PR China

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Abstract

The shapes of if-part fuzzy sets affect the approximating capability of fuzzy systems. In this paper, the fuzzy systems with the kernel-shaped if-part fuzzy sets are built directly from the training data. It is proved that these fuzzy systems are universal approximators and their uniform approximation rates can be estimated in the single-input–single-output (SISO) case. On the basis of these rates, the relationships between the approximating capability and the shapes of if-part fuzzy sets are developed for the fuzzy systems. Furthermore, the sinc functions that serve as input membership functions are proved to have the almost best approximation property in a particular class of membership functions. The theoretical results are confirmed from the simulation data. In addition, the estimations of the uniform approximation rates are extended to the multi-input–single-output (MISO) case.

Keywords: Fuzzy systems; Function approximation; Universal approximation; Uniform approximation rates; Almost best approximation

1. Introduction

Fuzzy systems are successfully used in many real areas [17,31,42] such as knowledge engineering, automatic control, pattern recognition, and so on. In order to improve their effectiveness in practical applications, the theoretical guidance for designing them must be further investigated [18,19]. Since Kosko and Wang both independently proved in 1992 that some fuzzy systems are universal approximators [15,41], researches on the approximation accuracy theory of fuzzy systems have attracted much attention [9,38]. The researchers deal mainly with three questions: (1) universal approximation; (2) constructive approximation; (3) approximation rates.

Recent studies show that various fuzzy systems have the universal approximation property [1,2,4–6,15,16,21–23,25,32,41,48,49,54]. For the fuzzy systems with the *and*/*or* operators, which can be depicted by

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* Corresponding author. Tel.: +86 731 4573245.
E-mail address: mrqiangluo@hotmail.com (Q. Luo).

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t-norms and t-conorms, respectively, Castro [5] proved that these fuzzy systems are capable of approximating any real continuous functions on a compact set to arbitrary accuracy. Yager and Kreinovich [48] proposed to use more general operators based on uninorms within the fuzzy systems modeling paradigm, and they also proved that these fuzzy systems are universal approximators. Some scholars also discussed the universal approximation properties of the hierarchical fuzzy systems [23,43]. Furthermore, some negative results have been obtained [13,24,36]: the fuzzy systems are nowhere dense in the space of continuous functions with respect to the supremum norm, if the number of rules is restricted on each input space. In one word, the question of the universal approximation has been well studied for the fuzzy systems.

Many scholars have studied the constructive approximation for various classes of fuzzy systems [10,14,40,44–47,50,51,53,54]. Ying [44] has presented the first result on this aspect; subsequently, Ying et al. [45–47] have obtained some sufficient conditions for fuzzy systems to approximate some continuous functions with the desired accuracy. Ding [10] has given some necessary conditions for the MISO Mamdani fuzzy systems to be the universal approximators. In addition, Zeng has established the approximation error bounds for the fuzzy systems generated by a center-average defuzzifier and also for the ones generated by an MoM defuzzifier with Pseudo-trapezoid-Shaped (PTS) membership functions [50,51]. That is, some guidelines have been proposed for the design of the aforementioned types of fuzzy systems.

However, there is still a lack of the theoretical estimations of the approximation rates of fuzzy systems [38]. The approximation rates play a key role in answering the question: what, if any, are the advantages of fuzzy systems as function approximators over the other methods such as the polynomial, spline, trigonometric, wavelets, neural networks. The function approximation with the fuzzy system is a kind of nonlinear approximation, in fact, a highly nonlinear approximation [7]. Given a target function f, the highly nonlinear approximation is supposed to choose both the fittest basis function from a class of basis functions and the best n-term approximation to f from this basis. In fuzzy systems, the basis functions are the if-part fuzzy sets. The systemic understanding of the highly nonlinear approximation is generally a big challenge [7]; specifically, the approximation rates of fuzzy systems have not been estimated so far, except for some particular type of fuzzy systems (e.g., the fuzzy KH interpolators [37]).

Some researchers explored the relationships between the shapes of the fuzzy sets (i.e., the membership functions) and the approximating capability of the fuzzy systems [18,19,52]. Zeng and Singh [52] have developed a relationship between the PTS membership functions and the accuracies of function approximation for fuzzy systems under some certain conditions. Mitaim and Kosko [18,19] put forward the open question: What is the best shape of a fuzzy set in fuzzy systems for function approximation. By exploring a wide range of candidate if-part sets, they found that no shape of the set emerges as the best shape; however, the numerical results show that the sinc function ($\sin x/x$) often converges fastest and with greatest accuracy among the candidates [19]. Unfortunately, they could not find any theoretical reason for the good performance of the sinc function as a nonlinear interpolator in a fuzzy system; however, they still suggest that an engineer should check whether choosing the sinc function to be the input membership function can improve a given fuzzy system.

Although the sinc-shaped fuzzy sets have many successful applications in both the numerical experiments [18,19,39] and the practical areas [20,26], some fundamental issues still need to be well addressed. For instance, the semantics of the sinc functions in the context of fuzzy systems and the approximation property of the sinc membership functions are of theoretical importance. For the semantics issue, the semantic integrity of a sinc membership function is discussed in the following paragraphs.

In comparison with the commonly used membership functions (e.g., the triangular membership function [28]) whose semantics are very clear in the context of fuzzy systems, the negative values and oscillatory nature of the sinc membership functions do not admit easy linguistic interpretation. Pedrycz et al. [27–31] have well discussed the requirements of the membership functions' semantic integrity that are deemed crucial to the development of the entire fuzzy model. The general concepts of the semantic integrity are briefly reviewed below [29]:

- *Distinguishability*: the membership functions of fuzzy sets should be defined on a certain range in the universe of discourse and be clearly distinguished from each other to represent some transparent semantic meanings.