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Issues on adjointness in multiple-valued logics

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Abstract

We contribute to the theory of implications and conjunctions related by adjointness, in multiple-valued logics. We suggest their use in Zadeh's compositional rule of inference, to interpret generalized modus ponens inference schemata. We provide new complete characterizations of implications that distinguish left arguments, implications that satisfy the exchange principle, divisible conjunctions, commutative conjunctions, associative conjunctions and triangular norms. We also introduce and characterize pseudo-strict and pseudo-continuous implications and conjunctions, and we explore the close relationship between these two notions.

Keywords: Non-classical logics; Implication; Conjunction; Adjointness; Generalized modus ponens

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1. Introduction

We are concerned with relating conjunctions to implications through adjointness, in (L, P)-valued propositional logic, whereby $(L, \leqslant)$ and $(P, \leqslant)$ always stand for two complete lattices, interpreting two types of *truth values*. In most applications, they are the unit interval of real numbers under its usual order. However, all proofs in this special case carry over, verbatim, to the general case of arbitrary complete lattices.

We begin with a synopsis of basic results on the notion of residuation (=adjointness) (Section 2). In Section 3, we propose using this notion in enlarging the scope of choice of connectives in Zadeh's compositional rule of inference. Then we supply new characterizations of implications that distinguish left arguments (Section 4), implications that satisfy the exchange principle (Section 5), divisible conjunctions (Section 6), pseudo-strict and pseudo-continuous implications and conjunctions (Section 7), commutative conjunctions (Section 8), associative conjunctions (Section 9) and triangular norms (Section 10).

2. Adjointness algebras

Throughout, $(P, \leqslant)$ and $(L, \leqslant)$ denote complete lattices. Their top elements are denoted by 1, and their bottom elements by 0. An implication A attaches to each ordered pair $(\alpha, \gamma) \in P \times L$ of partial truth values a partial truth value $A(\alpha, \gamma) \in L$. The basic intuitive demands on A are that the values $A(\alpha, \gamma)$ should be isotone in γ and antitone in α, and that A should satisfy the boundary conditions $A(0, 0) = A(1, 1) = 1$, in order that L-valued logic may subsume binary logic. We have also to obey the stipulation that A should preserve suprema in the left argument, and preserve infima in the right argument. This condition is mathematically necessitated by the need to simplify the solution sets of inequalities of the type $A(\alpha, \gamma) \geqslant \beta$, for either the unknown α or the unknown γ. It is also a necessary and sufficient condition for adjointness, see for instance [1]. These intuitive demands sum up in the following identity:

$$A\left(\sup_{j\in J} \alpha_j, \inf_{m\in M} \gamma_m\right) = \inf_{j,m} A(\alpha_j, \gamma_m) \tag{1}$$

for all subfamilies (possibly empty) $\{\alpha_j\}_{j\in J}$ of P and $\{\gamma_m\}_{m\in M}$ of L. Optionally, we may also consider one or both of the following two conditions on A:

$$\text{Neutrality axiom}: \quad \forall \gamma \in L: \quad A(1, \gamma) = \gamma, \tag{2}$$

$$\text{Comparator axiom}: \forall (\alpha, \gamma) \in L^2: \quad A(\alpha, \gamma) = 1 \text{ iff } \alpha \leqslant \gamma. \tag{3}$$