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Some constructions of the join of fuzzy subgroups and certain lattices of fuzzy subgroups with sup property

Naseem Ajmal [a], Aparna Jain [b,*]

[a] Department of Mathematics, Zakir Hussain College, University of Delhi, J.L. Nehru Marg, New Delhi, India
[b] Department of Mathematics, Shivaji College, University of Delhi, Ring Road, Raja Garden, New Delhi, India

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ABSTRACT

In this paper, some new lattices of fuzzy substructures are constructed. For a given fuzzy set μ in a group G, a fuzzy subgroup $S(\mu)$ generated by μ is defined which helps to establish that the set $\mathscr{L}_s$ of all fuzzy subgroups with sup property constitutes a lattice. Consequently, many other sublattices of the lattice $\mathscr{L}$ of all fuzzy subgroups of G like $\mathscr{L}_{s_t}$, $\mathscr{L}_{s_n}$, $\mathscr{L}_{s_{n_t}}$, etc. are also obtained. The notion of infimum is used to construct a fuzzy subgroup $i(\mu)$ generated by a given fuzzy set μ, in contrast to the usual practice of using supremum. In the process a new fuzzy subgroup $i(\mu)^*$ is defined which we shall call a shadow fuzzy subgroup of μ. It is established that if μ has inf property, then $i(\mu)^*$ also has this property.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [32], with a view to apply it in approximate reasoning. The wide applications of fuzzy set theory and fuzzy logic are well known by now. The theory has found very useful applications in diverse fields such as automata, control theory, decision analysis, social behaviour pattern studies, to name just a few. Also, the theory of groups occupies a significant place in Mathematics, as well as outside with many practical applications in, for example nuclear physics and information sciences. Therefore, an amalgamation of these two theories was a natural evolution. Rosenfeld [28], in 1971, initiated the studies of fuzzy group theory.

On the other hand, lattice theory has well known significant applications in various branches of human knowledge systems including information sciences. A lattice theoretic approach in the studies of neural networks is employed by many researchers. A detailed account of applications of lattice theory in computational intelligence are provided in [19] by Kaburlasos and Ritter. For example, Kaburlasos himself discusses granular enhancement of fuzzy neural classifiers based on lattice theory; while Ritter and Urcid studied learning in lattice neural networks that employ Dentritic computing. Moreover, Barmpoutis and Ritter studied the orthonormal basis lattice neural networks. The role of lattice theory is further highlighted in parallel distributed concept learning and visual pattern recognition. A lattice based approach to mathematical morphology for greyscale and colour images is studied by Kaburlasos and Ritter in [19]. The emergence of lattice theory within the field of computational intelligence is partially due to its proven effectiveness in neural computation. Moreover, lattice theory has the potential to unify a number of diverse concepts and aid in the cross fertilization of both tools and ideas within the numerous

* Corresponding author. Address: D-184, Ashok Vihar, Phase I, Delhi 110052, India. Tel.: +91 11 42271500.
E-mail addresses: nasajmal@yahoo.com (N. Ajmal), jainaparna@yahoo.com (A. Jain).

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subfields of computational intelligence. In [19], it is very well demonstrated that how lattice theory may suggest viable alternatives in practical clustering, classification, pattern analysis and regression application.

That the system of lattice algebra plays a significant role in information theory was also demonstrated in [11]. In [13], Carpineto and Romano applied lattices to information retrieval. In [29], Sandhu showed that lattice based mandatory access controls can be enforced by appropriate configuration of RBAC components. Applications of lattice theory to graph decomposition was studied by Liang et al. in [21].

Moreover, as a kind of important non-classical logic, lattice valued logic provides facilities to describe and deal with information or knowledge with fuzziness and incomparability. In order to establish an alternative logic for knowledge representation and reasoning, Xu [31] proposed a logical algebra; that is, lattice implication algebra, in 1993 by combining algebraic lattice and implication algebra. The applications of lattice implication algebras in machine intelligence have been considered by Ma et al. in [23]. In [35], it has been shown that the family of all formal concepts with reference to the context form a complete lattice, called the concept lattice. As an effective tool for data analysis, formal concept analysis has been extensively applied to fields such as decision making, information retrieval, data mining and knowledge discovery. From the point of view of fuzzy logic, Burusco and Fuentes-Gonzales [12] developed concept lattice theory in the fuzzy environment.

In [3], the notion of fuzzy lattices was introduced. This concept has been employed very effectively in [17] to study fuzzy lattice reasoning classifiers and their application for ambient ozone estimation source. Moreover, the concept of fuzzy lattices has been used to solve several engineering and computer science problems [18,26,27]. In [18], fuzzy lattice neurocomputing models (FLN) have been constructed. In [26], fuzzy lattice neural networks: A hybrid model for learning are developed. Further, in [27], learning in the framework of fuzzy lattices is studied.

In view of the overwhelming applications of lattices in modern information sciences, we construct here some new lattices of fuzzy subgroups. Ajmal and Thomas initiated such studies in the realm of fuzzy group theory and did extensive work in this field [1–9]. It was shown in [2] that $\mathscr{L}$, the set of all fuzzy subgroups of a group, constitutes a lattice under the ordering of fuzzy set inclusion. It was also demonstrated in [2] that $\mathscr{L}_t$, the set of all fuzzy subgroups of a group G with the same tip 't' forms a complete sublattice of the lattice $\mathscr{L}$. Various other sublattices of the lattice $\mathscr{L}$ have been constructed by Ajmal and Thomas in [2,5,8]. One of these is $\mathscr{L}_{f_{n_t}}$, the set of all fuzzy normal subgroups of a group G each of which has finite range set and a fixed tip 't'. This set $\mathscr{L}_{f_{n_t}}$ was shown to be a modular sublattice of the lattice $\mathscr{L}$. Then, it was shown in [6] that $\mathscr{L}_n$, the set of all fuzzy normal subgroups of a group is a modular sublattice of $\mathscr{L}$. Later, in [15], the author provided a much simpler and direct proof of modularity of the lattice of fuzzy normal subgroups. In [16], the authors exhibited the method to construct the lattices of the sets of all fuzzy subsets (fuzzy subgroups) of a fuzzy set (fuzzy group) in the newly defined categories $\mathscr{S}$ and $\mathscr{G}$ of fuzzy sets and fuzzy groups respectively.

The notion of sup property was introduced by Rosenfeld in [28] to obtain certain results. This property turned out to be an important concept in fuzzy algebra and was successfully used in extending numerous results of classical algebra into fuzzy setting. The concept of sup property has very deep and pleasing consequences in the sense that it helps in extending results of crisp algebra to fuzzy algebra on one hand and on the other hand, this property of functions gives rise to the construction of new lattices which is a peculiarity of fuzzy setting.

A thorough study of fuzzy subgroups with sup property was done by Ajmal in [8] wherein the inner structure of such fuzzy subgroups was unveiled and their characterization in terms of level subsets was provided. Moreover in that paper, the author studied the algebraic nature of homomorphic images and preimages of a fuzzy subgroup satisfying this property. It was also shown in [8] that Liu's sup min product $\mu \circ \eta$ of two fuzzy subgroups μ and η with sup property also has sup property. Then, using $\mu \circ \eta$ as the join of μ and η and $\mu \cap \eta$ as their meet, $\mathscr{L}_{n_{s_t}}$, the set of all fuzzy normal subgroups of a group G with sup property and having the same tip 't' was shown to be a sublattice of $\mathscr{L}_{n_t}$, which is the set of all fuzzy normal subgroups of G having the same tip 't'. Readers are also referred to Chapter 9 of [25] and Chapter 1 of [24] for the theory of the lattices of fuzzy subgroups and fuzzy subgroups with sup property. Various aspects of this peculiar construction of fuzzy subgroups with sup property have been discussed in these chapters.

In this paper, some more new lattices of fuzzy subgroups are constructed. This is achieved by slightly modifying the construction of join structure of fuzzy subgroups. Sultana and Ajmal [30] defined a fuzzy subgroup μ^* generated by a given fuzzy set μ as follows:

$$\mu^*(x) = \sup_{t \leqslant \sup \mu} \{t / x \in \langle \mu_t \rangle\}.$$

In the present paper, this construction is altered and we take supremum over $\operatorname{Im} \mu$ (see Definition 3.3) to obtain a fuzzy subgroup $S(\mu)$ of the group G and prove that this fuzzy subgroup $S(\mu)$ is the fuzzy subgroup generated by the given fuzzy set μ. This provides a better formulation of the join structure of two fuzzy subgroups of a group. Using this construction, it has been established that $\mathscr{L}_s$, the set of all fuzzy subgroups of G with sup property constitutes a lattice, which is a sublattice of $\mathscr{L}$. Many other sublattices of the lattice $\mathscr{L}$ like $\mathscr{L}_f, \mathscr{L}_t, \mathscr{L}_n, \mathscr{L}_{n_t}, \mathscr{L}_{s_t}$ and $\mathscr{L}_{f_t}$ are also obtained.

In Section 4, the notion of supremum is replaced by infimum along with some other suitable changes to formulate another fuzzy subgroup $i(\mu)^*$ which we call a shadow fuzzy subgroup of the given fuzzy set μ. Although, $i(\mu)^*$ is not the fuzzy subgroup generated by μ, which is illustrated in Example 4.6, it possesses some interesting properties which are worth investigating further. It has been established that if μ has inf property, then $i(\mu)^*$ has inf property. Then, the definition of $i(\mu)^*$ is modified further to get another fuzzy set $i(\mu)$ (Definition 4.10), which turns out to be the fuzzy subgroup generated by the fuzzy set μ. However, it could not be established that $i(\mu)$ has inf property whenever μ has inf property. Had it