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Maximum entropy membership functions for discrete fuzzy variables

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ABSTRACT

Due to the deficiency of information, the membership function of a fuzzy variable cannot be obtained explicitly. It is a challenging work to find an appropriate membership function when certain partial information about a fuzzy variable is given, such as expected value or moments. This paper solves such problems for discrete fuzzy variables via maximum entropy principle and proves some maximum entropy theorems with certain constraints. A genetic algorithm is designed to solve the general maximum entropy model for discrete fuzzy variables, which is illustrated by some numerical experiments.

1. Introduction

Fuzziness is an usual uncertainty in the real world, which results from the vague concepts due to indeterminate boundaries [29] or arises in logic [30] when a proposition can be treated as neither definitely true nor definitely false. In order to deal with the behavior of fuzzy phenomena, Zadeh [26] introduced the concept of fuzzy set via the membership function in 1965, and then proposed possibility measure as a mathematical framework in 1978 (see [28]). Possibility theory was developed by many researchers such as Nahmias [16], Yager [21], De Cooman [2], Dubois and Prade [4], and became a powerful tool to deal with fuzziness. However, possibility measure has no self-duality property. To make up this drawback, Liu and Liu [10] introduced a self-dual measure, called *credibility measure*, to quantify the chance of occurrence of fuzzy events. Also, Li and Liu [8] gave a sufficient and necessary condition for credibility measure in 2006. On the basis of the credibility measure, the credibility theory has been founded by Liu [11] as a branch of mathematics to study the behavior of fuzziness. For more details of credibility theory, interested readers may refer to Liu [12,14].

Fuzzy entropy is a term used to represent the degree of uncertainty. It was first quantified as a weighted Shannon entropy based on a probabilistic framework in 1968 by Zadeh [27]. De Luca and Termini [3] defined a nonprobabilistic fuzzy entropy based on Shannon function as a measure of fuzziness in 1972. In 1975, Kaufmann [5] suggested that the fuzziness of a fuzzy set can be measured through the distance between the fuzzy set and its nearest set. After that, Knopfmacher [6] and Loo [15] extended the definition of fuzzy entropy made by De Luca and Termini, and Kaufmann, respectively. In 1979, Yager [20,21] introduced another kind of fuzzy entropy measure by using the distance of the fuzzy set and its complement. Two new measures of weighted fuzzy entropy were introduced by Parkasha et al. [19]. Besides these, a lot of research works have been done concerning fuzzy entropy and its applications such as Yager [22–24], Pal and Pal [17,18], Bhandari and Pal [1]. Based on the credibility theory, Li and Liu [7] introduced a kind of fuzzy entropy to measure the degree of uncertainty associated

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with fuzzy variables. You and Wen [25] proposed a similar definition of entropy for fuzzy vectors. A detailed survey of entropy of fuzzy variables can be found in Liu [13].

For practical problems without sufficient information, the maximum entropy principle is a widely accepted method to determine uncertainty distributions. For a fuzzy variable, in some circumstances, we cannot get the membership function explicitly, while certain partial information such as the expected value and variance may be obtained easily. There are infinitely many membership functions consistent with such partial information. Which membership function should we take? Following the maximum entropy principle, we will select the membership function which maximizes the value of entropy and satisfies the prescribed constraints. The selected function is called the maximum entropy membership function, denoted by MEMF. For continuous fuzzy variables, Li and Liu [9] gave an analytical method to find the MEMF. In this paper, we modify Li and Liu's idea and give out our method to solve this problem for discrete fuzzy variables. The organization of our work is as follows: In Section 2, some basic concepts and results on fuzzy variable are reviewed. In Section 3, we introduce some constraints on the expected values, and then deduce the possible MEMF. In Section 4, the MEMF given an inequality constraint on the expected value is explored. In Sections 5 and 6, an effective genetic algorithm is introduced to solve general maximum entropy models for discrete fuzzy variables and some computational experiments are given in illustration of it. Finally, the conclusion is given in the last section.

2. Preliminaries

Let ξ be a fuzzy variable with the membership function $\mu(x)$ which satisfies the normalization condition, i.e., $\sup_x \mu(x) = 1$. In the setting of credibility theory, the credibility measure for fuzzy event $\{\xi \in B\}$ deduced from $\mu(x)$ is given by

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right),$$

where B is any subset of the real numbers $\Re$, and B^c is the complement of set B. Conversely, for a fuzzy variable ξ, its membership function can be derived from the credibility measure by

$$\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re.$$

The expected value of ξ was defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geqslant r\}\,\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leqslant r\}\,\mathrm{d}r$$

provided that at least one of the two integrals is finite. If ξ is a discrete fuzzy variable taking values in $\{x_1, x_2, \ldots, x_n\}$ (in this paper we always assume that $x_1 < x_2 < \cdots < x_n$) with membership degree $\{\mu_1, \mu_2, \ldots, \mu_n\}$, where $\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n = 1$, then the expected value can be written as (without loss of generality, suppose $x_{k-1} < 0 \leqslant x_k$)

$$\begin{aligned}
E[\xi] &= \int_0^{x_k} \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r + \sum_{i=k+1}^{n}\int_{x_{i-1}}^{x_i} \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r - \sum_{i=2}^{k-1}\int_{x_{i-1}}^{x_i} \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r - \int_{x_{k-1}}^{0} \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r \\
&= \sum_{i=1}^{k-1}(\mathrm{Cr}\{\xi \leqslant x_i\} - \mathrm{Cr}\{\xi < x_i\})\cdot x_i + \sum_{i=k}^{n}(\mathrm{Cr}\{\xi \geqslant x_i\} - \mathrm{Cr}\{\xi > x_i\})\cdot x_i \\
&= \sum_{i=1}^{n}\frac{1}{2}\left(\max_{1\leqslant j\leqslant i}\mu_j - \max_{1\leqslant j<i}\mu_j + \max_{i\leqslant j\leqslant n}\mu_j - \max_{i<j\leqslant n}\mu_j\right)\cdot x_i \equiv \sum_{i=1}^{n}\omega_i x_i, \qquad (1)
\end{aligned}$$

where the weights are given by

$$\omega_i = \frac{1}{2}\left(\max_{1\leqslant j\leqslant i}\mu_j - \max_{1\leqslant j<i}\mu_j + \max_{i\leqslant j\leqslant n}\mu_j - \max_{i<j\leqslant n}\mu_j\right) \qquad (2)$$

for $i = 1, 2, \ldots, n$. It is easy to verify that all $\omega_i \geqslant 0$ and $\sum_{i=1}^{n}\omega_i = 1$. Furthermore, $E\left[(\xi - e)^2\right]$ is called the variance of ξ and $E[\xi^n]$ the nth moment of ξ.

Li and Liu [7] introduced a new definition of entropy of a fuzzy variable ξ, denoted by $H[\xi]$. For a discrete fuzzy variable ξ taking values in $\{x_1, x_2, \ldots, x_n\}$,

$$H[\xi] = \sum_{i=1}^{n} S(\mathrm{Cr}\{\xi = x_i\}), \qquad (3)$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$ with the convention that $S(0) = 0$. It is obvious that $0 \leqslant H[\xi] \leqslant n\ln 2$. The left-hand side equality holds if and only if ξ is essentially a crisp number, and the right-hand side equality holds if and only if $\mathrm{Cr}\{\xi = x_i\} = 0.5$, i.e., $\mu(x_i) \equiv 1$ for all $i = 1, 2, \ldots, n$.

For a continuous fuzzy variable ξ, $H[\xi] = \int_{-\infty}^{+\infty} S(\mathrm{Cr}\{\xi = x\})\mathrm{d}x$. By using maximum entropy principle, it's easy to obtain the following results [9]: if ξ is nonnegative with a finite second moment m^2, $H[\xi] \leqslant \pi m/\sqrt{6}$ with equality when