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Axiomatization and conditions for neighborhoods in a covering to form a partition ☆

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ABSTRACT

In this paper, we study the axiomatic issue of a type of covering upper approximation operations. This issue was proposed as an open problem. We also further some known results by using only a single covering approximation operator to characterize the conditions for neighborhood $\{N(x)\colon x \in U\}$ to form a partition of universe U.

1. Introduction

The classical rough set theory is built on equivalence relation [4,10–16,18,23,28]. However, equivalence relation imposes restrictions and limitations on many applications [20,21,30,31]. Zakowski then established the covering-based rough set theory by exploiting coverings of a universe [24]. This brings various covering approximation operators [3,19,22]. Axiomatizations of these covering approximation operators are important topics in covering-based rough set theory [7,17,27–30,32]. As a type of special coverings, topology provides many useful techniques in covering-based rough set theory [1,5,6,8,9]. From the perspective of point-set topology, the core concept of covering approximation operators is neighborhood $N(x)$ of a point x. Therefore, it is important to study the conditions for $\{N(x)\colon x \in U\}$ to form a partition of universe U [17].

In this paper, we construct a set of axioms to characterize a class of covering upper approximation operations for an arbitrary universe. In particular, when the universe is finite, our results provide answers to an open problem proposed by W. Zhu and F. Wang in paper [32]. We also further some results presented by K. Qin et al. in paper [17] by providing sufficient and necessary conditions for $\{N(x)\colon x \in U\}$ to form a partition on U.

Before our discussion, we introduce some basic concepts used in this paper. U always denotes a non-empty arbitrary set unless it is specially mentioned. C denotes a covering of U and $P(U)$ denotes the family of subsets of U. We call ordered pair (U,C) a covering approximation space, and $N(x) = \cap\{K \in C\colon x \in K\}$ neighborhood of point x for each $x \in U$.

Let (U,C) be a covering approximation space. Our discussion in this paper involves six types of covering approximation operations that are defined as follows: for $X \subseteq U$,

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(1) $\underline{C_1}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subseteq X\}, \overline{C_1}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\}$;
(2) $\underline{C_2}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subseteq X\}, \overline{C_2}(X) = U - \underline{C_2}(U - X)$;
(3) $\underline{C_3}(X) = \{x \in U : N(x) \subseteq X\}, \overline{C_3}(X) = \{x \in U : N(x) \bigcap X \neq \emptyset\}$;
(4) $\underline{C_4}(X) = \{x \in U : \exists u(u \in N(x) \wedge N(u) \subseteq X)\}, \overline{C_4}(X) = \{x \in U : \forall u(u \in N(x) \to N(u) \cap X \neq \emptyset)\}$;
(5) $\underline{C_5}(X) = \{x \in U : \forall u(x \in N(u) \to N(u) \subseteq X)\}, \overline{C_5}(X) = \bigcup\{N(x) : x \in U \bigwedge N(x) \bigcap X \neq \emptyset\}$; and
(6) $\underline{C_6}(X) = \{x \in U : \forall u(x \in N(u) \to u \in X)\}, \overline{C_6}(X) = \bigcup\{N(x) : x \in X\}$.

Remark 1.1. We call $\underline{C_n}$ the covering lower approximation operation and $\overline{C_n}$ the covering upper approximation operation (n = 1,2,3,4,5,6). $\underline{C_1}$ and $\overline{C_1}$ come from [32], and for $2 \leqslant n \leqslant 6$, $\underline{C_n}$ and $\overline{C_n}$ come from [17].

Let $1 \leqslant n \leqslant 6$. If for each $X \subseteq U$, approximation operators $\underline{C_n}$ and $\overline{C_n}$ satisfy $\underline{C_n}(X) = -\overline{C_n}(-X)$, then $\underline{C_n}$ and $\overline{C_n}$ are called dual approximation operators. Clearly, except $\underline{C_1}$ and $\overline{C_1}$, all other $\underline{C_n}$ and $\overline{C_n}$ are dual approximation operators.

The rest sections of this paper are arranged as follows. In Section 2, we construct a set of axioms that characterizes a covering upper approximation operation defined in [32] and explore the properties of this operator from the perspective of point-set topology. In Section 3, we use a single covering approximation operator to give the conditions for $\{N(x): x \in U\}$ to form a partition of U. Section 4 concludes the paper.

2. Axiomatization and topological properties of covering approximation operation C_1

Axiomatization of covering upper approximation operation $\overline{C_1}$ for a non-empty finite set U was proposed as an open problem [32]. In the following, we present an axiomatization theorem of $\overline{C_1}$ for any nonempty arbitrary U.

Theorem 2.1. *An operation L: $P(U) \to P(U)$ satisfies the following properties: for any X, $Y \subseteq U$,*

(1) $\forall x \in U$, $x \in L(\{x\})$,
(2) $L(\emptyset) = \emptyset$,
(3) $\forall H \subseteq U$, $L(H) = \cup_{x \in H} L(\{x\})$, *and*
(4) $\forall x, y \in U$, $x \in L(\{y\}) \iff y \in L(\{x\})$,

if and only if there exists a covering C of U, such that covering upper approximation operation $\overline{C_1}$ generated by C is equal to L.

Proof. Sufficiency is directly from the definition of $\overline{C_1}$. We only prove necessity. We construct a covering of U as follows. For each $x, y \in U$, where $x = y$ or $x \neq y$, define $V(x,y) = \cap\{L(z): x, y \in L(z)\}$, and $C = \{V(x,y): x, y \in U$, and $x \in L(y)\}$. By property (1) of L, we have $x \in V(x,x) \in C$, and hence C is a covering of U. Now we prove that covering upper approximation operation $\overline{C_1}$ generated by C is equal to L, that is, $\forall H \subseteq U, L(H) = \overline{C_1}(H)$. If $H = \emptyset$, then $L(H) = \overline{C_1}(H)$ because of property (2) of L; if $H \neq \emptyset$, then for each $x \in L(H)$, by property (3) of L, there exists $y \in H$ such that $x \in L(\{y\})$. By property (4) of L, $y \in L(\{x\})$. From the definition of $V(x,y)$, we have $x, y \in V(x,y) \in C$ and $V(x,y) \cap H \neq \emptyset$. Thus, it follows that $x \in \overline{C_1}(H)$. So we have $L(H) \subseteq \overline{C_1}(H)$ from arbitrariness of x. On the other hand, since both $\overline{C_1}$ and L satisfy property (3), it is enough to prove that: $\forall x \in H, \overline{C_1}(\{x\}) \subseteq L(\{x\})$. Since $\overline{C_1}(\{x\}) = \cup\{V(y,z) : y, z \in U, y \in L(z) \wedge x \in V(y,z)\}$ and $x \in \overline{C_1}(\{x\})$, $x \in L(y) \cap L(z)$. By property (4) of L, we have y, $z \in L(x)$. From the definition of $V(y,z)$, we know that $V(y,z) \subseteq L(\{x\})$. So we have $\overline{C_1}(\{x\}) \subseteq L(\{x\})$. Thus, we have proved that $L(H) = \overline{C_1}(H)$. □

Remark 2.2. The proof technique in Theorem 2.1 was used in two papers in the area of point-set topology by Balogh et al. [1] and Nagata [9].

If U is a finite set and there is a topology on set U, then operation $\overline{C_1}$ constructed in Theorem 2.1 has the following topological property.

Theorem 2.3. *Let U be a finite set and L, C be defined as in Theorem 2.1. If there is a topology on set U, then $\forall X \subseteq U$, $L(X)$ is an open set if any only if C is an open covering.*

Proof. For every $X \subseteq U$, suppose $L(X)$ is an open set. Since U is a finite set, $\{L(x): x \in U\}$ is also finite. By construction of covering C, every element in C is an intersection of finite elements in set $\{L(x): x \in U\}$. Then C is an open covering since $\{L(x): x \in U\}$ is a collection of open sets and the intersection of finite open sets is still an open set.

On the other hand, for every $X \subseteq U$, since C is an open covering, $\overline{C_1}(X)$ is also an open set. By Theorem 2.1, we know that covering upper approximation operation $\overline{C_1}$ generated by C is equal to L, so $L(X)$ is an open set for every $X \subseteq U$. □

Remark 2.4. Zhang et al. in [26] also gave the same set of axioms for $\overline{C_1}$ as we did in Theorem 2.1. However, covering C that is constructed in [26] is different from the covering constructed in this paper. Covering $C = \{\{x,y\}: y \in L(\{x\}), x, y \in U\}$ constructed in [26] is a family consisting of two-point sets. Theorem 2.3 does not always hold for such a covering. This constrains