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Blackwell's Theorem for *T*-related fuzzy variables

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ABSTRACT

In this paper, we consider Blackwell's Theorem in which inter-arrival times are characterized as fuzzy variables under *t*-norm-based fuzzy operations. We first prove that Blackwell's Theorem for *T*-related fuzzy variables with respect to necessity measure holds true where *T* is an Archimedean *t*-norm. Subsequently, we provide a counter example under which Blackwell's Theorem does not hold when $T = min$. Finally, we evaluate the expected value of fuzzy variable with respect to credibility measure and derive fuzzy Blackwell's Theorem based on the expected value of fuzzy variables.

1. Introduction

The theory of fuzzy sets introduced by Zadeh [18–21] has been extensively studied and applied to the area of statistics and possibility areas in recent years. Since Puri and Ralescu [15] introduced the concept of fuzzy random variables, there has been an increasing interest in fuzzy variables. But there have been only a few papers [9,10,14,20] investigating the renewal theory in fuzzy environment. Hwang [10] considered the stochastic process for fuzzy random variables and proved a theorem for the fuzzy rate of a fuzzy renewal process. Popova and Wu [14] proposed a theorem presenting the long-run average fuzzy reward by using the strong law of large numbers. Zhao and Liu [22] and Hong [9] discussed the renewal process under the consideration of fuzzy inter-arrival times and proved that the expected reward per unit time is the expected value of the ratio of the reward spent in one cycle to the length of the cycle. Zhao and Tang [23] obtained some properties of fuzzy random renewal process generated by a sequence of independent and identically distributed fuzzy random interval times based on fuzzy random theory, especially Blackwell's Theorem for fuzzy random variables. They all utilized *min*-norm-based fuzzy operations. In general we can consider the extension principle realized by the means of some *t*-norm. The renewal theory is closely related with the law of large numbers. Many different types of the law of large numbers for *T*-related fuzzy variables have been studied by a number of authors, such as, Badard [1], Fullér [3], Triesch [17], Marková [13], Hong [4], Hong and Lee [6], Hong and Ro [5], and Hong and Ahn [7]. Recently, Hong [8] considered a renewal reward process in which inter-arrival times and rewards are modeled as fuzzy variables using *t*-norm-based fuzzy operations. And Hong [8] proposed a fuzzy renewal theorem and a fuzzy renewal reward theorem through necessity measure. In this paper, we evaluate further fuzzy versions of Blackwell's Theorem based on necessity measure and expected value of fuzzy variables.

In Section 2, we provide definitions and basic results of *T*-sum of *L*–*R* fuzzy variables. In Section 3, we discuss renewal processes and Blackwell's Theorem under the consideration of fuzzy inter-arrival times. We propose a fuzzy version of Blackwell's Theorem based on necessity measure for the rate of the renewal process having fuzzy inter-arrival times. In Section 4, we evaluate a fuzzy renewal theorem and the fuzzy version of Blackwell's Theorem based on expected value of fuzzy variables.

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2. Preliminaries

Let $(\Gamma, P(\Gamma), Pos)$ be a possibility space. As defined in [2], a normal fuzzy variable is defined as a function $\xi : \Gamma \to R$ which has a unimodal, upper semi-continuous membership function μ_ξ on the real line such that there exists a unique real number m satisfying $\mu_\xi(m) = \sup_x \mu_\xi(x) = 1$. The number $m = m(\xi)$ is called the modal value of ξ. Then, let us assume that a sequence of fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ and a t-norm T [24] are given. The T-sum $\xi_1 + \xi_2 + \cdots + \xi_n$ and corresponding T-arithmetic mean $(\xi_1 + \xi_2 + \cdots + \xi_n)/n$ are the fuzzy variables as defined by

$$\mu_{\xi_1+\xi_2+\cdots+\xi_n}(z) := \sup_{x_1+x_2+\cdots+x_n=z} T(\mu_{\xi_1}(x_1), \ldots, \mu_{\xi_n}(x_n))$$

and

$$\frac{1}{n}(\xi_1 + \xi_2 + \cdots + \xi_n)(z) := (\xi_1 + \xi_2 + \cdots + \xi_n)(nz),$$

respectively [2]. For a fuzzy variable ξ and any subset D of the real numbers, the quantity

$$Nes(\xi|D) := 1 - \sup_{x \notin D} \mu_\xi(x) := 1 - Pos(\xi|D^c)$$

is considered to measure the necessity of ξ belonging to D (see [18]). If D is an interval (a,b), we may also write $Nes(a < \xi < b)$ instead of $Nes(\xi|D)$.

Assume that a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ of fuzzy variables and a t-norm T are given and denoted by m_n the modal value of the T-arithmetic mean $(\xi_1 + \xi_2 + \cdots + \xi_n)/n$. Following Fullér [3], we say that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ obey the law of large numbers if for all $\epsilon > 0$ the quantity $Nes(m_n - \epsilon < (\xi_1 + \xi_2 + \cdots + \xi_n)/n < m_n + \epsilon)$ tends to 1 as $n \to \infty$. It is noted that for any $\epsilon > 0$,

$$\lim_{n\to\infty} Nes(m_n - \epsilon < (\xi_1 + \xi_2 + \cdots + \xi_n)/n < m_n + \epsilon) = 1$$

if and only if

$$\lim_{n\to\infty} Nes(m_n - \epsilon \leqslant (\xi_1 + \xi_2 + \cdots + \xi_n)/n \leqslant m_n + \epsilon) = 1.$$

Recall that a t-norm T is Archimedean if and only if $T(x,x) < x$ for all $x \in (0,1)$. A well-known theorem [16] asserts that for each continuous Archimedean t-norm there exists a continuous, decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ such that

$$T(x_1, \ldots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n))$$

for all $x_i \in (0,1),\ 1 \leqslant i \leqslant n$. Here $f^{[-1]} : [0,\infty] \to [0,1]$ is defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{for } y \in [0, f(0)], \\ 0 & \text{if } y > f(0). \end{cases}$$

The function f is called the additive generator of T. Notice that if a continuous t-norm T has an additive generator f, then this additive generator is uniquely determined up to a non-zero positive multiplicative constant.

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and decreasing, and we have

$$\mu_{\xi_1+\cdots+\xi_n}(z) = \sup_{x_1+\cdots+x_n=z} f^{[-1]}\left(\sum_{i=1}^n f(\xi_i(x_i))\right) = f^{[-1]}\left(\inf_{x_1+\cdots+x_n=z}\left(\sum_{i=1}^n f(\xi_i(x_i))\right)\right).$$

An L–R fuzzy variable $\xi = (a, \alpha, \beta)_{LR}$ has a membership function from the reals into interval [0,1] satisfying

$$\mu_\xi(t) = \begin{cases} R\left(\frac{t-a}{\beta}\right) & \text{for } a \leqslant t \leqslant a + \beta, \\ L\left(\frac{a-t}{\alpha}\right) & \text{for } a - \alpha \leqslant t \leqslant a, \\ 0 & \text{otherwise}, \end{cases}$$

where L and R are non-increasing and continuous functions from [0,1] to [0,1] satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

Marková [13] obtained the following result of the law of large numbers for fuzzy variables.

Theorem 1 [13]. *Let $\xi_i = (a, \alpha_i, \beta_i)_{LR},\ i = 1, 2, \ldots$, denote a sequence of L–R fuzzy variables with bounded spread sequences, i.e., $\alpha_i \leqslant K,\ \beta_i \leqslant K,\ i \in N$ for some $K > 0$. Then the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ obeys the law of large numbers with respect to any continuous Archimedean t-norm T.*

3. Blackwell's Theorem w.r.t. necessity measure

Let ξ_n denote the times between the $(n-1)$th and the nth events, known as the inter-arrival times, where $n = 1, 2, \ldots$, respectively. Define

$$S_0 = 0, \quad S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \quad n \geqslant 1.$$