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Distributivity in lattices of fuzzy subgroups ☆

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ABSTRACT

The main goal of this paper is to study the finite groups whose lattices of fuzzy subgroups are distributive. We obtain a characterization of these groups which is similar to a well-known result of group theory.

1. Introduction

One of the most important facets of fuzzy logic is constitute by fuzzy set theory (see [29]). This topic has enjoyed a rapid evolution in the last years (for example, see [7]). It is an ingredient in the development of information technologies: information processing, information systems, Artificial Intelligence, and so on. Applications have also been found in industrial engineering [8], in decision and organization sciences (preference modeling and multicriteria evaluation), medical sciences (fuzzy expert systems) or in many other disciplines pertaining to human-oriented studies such as cognitive psychology and some aspects of social sciences. Several new research directions appeared in connection with mathematical branches: category theory, topology, algebra, analysis or probability theory (especially (GTU) – see [28]).

In algebra, a fundamental domain is the lattice theory. The formation of lattices is an important feature of many structures such as subgroups of a group, ideals of a ring, submodules of a module, or ideals of a lattice. Consequently, the study of some basic properties of these structures can be carried out within the purview of lattice theory. One of the most famous examples that illustrate this fact is concerning to finite groups and due to Ore [19]: *a finite group is cyclic if and only if its lattice of subgroups is distributive*.

The concept of fuzzy subgroup of a group has been introduced by Rosenfeld [20]. After 1971, an important step in fuzzy subgroup theory was made by Liu [12], which formulated the notion of fuzzy normal subgroup. This is closely connected to that of fuzzy congruence, studied in [2,4,9,11], or [13]. The set $FL(G)$ of all fuzzy subgroups of a given group G forms a complete lattice under the usual ordering of fuzzy set inclusion (see [1]). Several remarkable sublattices of $FL(G)$ have been investigated in [1–3]. From these, we mention the lattice $FN(G)$ of all fuzzy normal subgroups which is modular, as show [3], or [6]. Also, recall here the technique initiated in [6] (see [27], too) to derive fuzzy theorems from their crisp versions.

Another significant direction in the above study is to classify the fuzzy subgroups of a finite group. This can be made by introducing some natural equivalence relations on its lattice of fuzzy subgroups. Many papers have treated the particular

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case of finite cyclic groups. Thus, in [15] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [16–18] and [26] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). Remind also the paper [25] where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups of an arbitrary finite cyclic group. It plays an essential role in establishing of an explicit formula for the well-known central Delannoy numbers (see [24]).

The distributivity constitutes a very powerful property of a lattice. In this paper it is studied for the lattice $FL(G)$ of fuzzy subgroups of a finite group G. We shall use a new equivalence relation on $FL(G)$ that permits us to link this lattice to the lattice of subgroups of G. In this way, a result similar to Ore's theorem will be obtained.

The paper is organized as follows: in Section 2 we present some preliminary definitions and results on the subgroups and fuzzy subgroups of (finite) groups. Section 3 deals with a connection between the subgroup lattice $L(G)$ and the fuzzy subgroup lattice $FL(G)$ associated to a group G. Section 4 is dedicated to proving our main theorem, that characterizes the finite groups G for which $FL(G)$ is a distributive lattice. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on groups (respectively on fuzzy groups) can be found in [22] (respectively in [10]). For subgroup lattice concepts we refer the reader to [21,23].

2. Preliminaries

Let $(G, \cdot, e)$ be a group, where e denotes the identity of G. Then the set $L(G)$ of all subgroups of G is a complete bounded lattice with respect to set inclusion, called the *subgroup lattice* of G. Note that $L(G)$ has initial element the trivial subgroup $\{e\}$ and final element G, and its binary operations $\wedge, \vee$ are defined by

$$H \wedge K = H \cap K,\ H \vee K = \langle H \cup K \rangle \quad \text{for all } H, K \in L(G).$$

A remarkable modular sublattice of $L(G)$ is the *normal subgroup lattice* $N(G)$, which consists of all normal subgroups in G. Many results concerning to the relation between the structure of a group and the structure of its (normal) subgroup lattice are presented in [21,23]. We recall here only the following beautiful theorem, constituting the starting point for our discussion.

Theorem. (Ore [19]) *A group G is locally cyclic if and only if $L(G)$ is a distributive lattice. In particular, a finite group G is cyclic if and only if $L(G)$ is a distributive lattice.*

In the following, let us denote by $\mathscr{F}(G)$ the collection of all fuzzy subsets of G (which is a complete and completely distributive lattice under the usual intersection, union and containment of fuzzy sets) and take an element μ of $\mathscr{F}(G)$. Then μ is said to be a *fuzzy subgroup* of G if it satisfies the next two conditions:

(a) $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;
(b) $\mu(x^{-1}) \geqslant \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \sup \mu(G)$. If μ satisfies the supplementary condition

$$\mu(xy) = \mu(yx) \quad \text{for all } x, y \in G,$$

then it is called a *fuzzy normal subgroup* of G. As in the case of subgroups, the sets $FL(G)$ and $FN(G)$ consisting of all fuzzy subgroups and all fuzzy normal subgroups of G form lattices with respect to fuzzy set inclusion (more precisely, $FN(G)$ is a sublattice of $FL(G)$), called the *fuzzy subgroup lattice* and the *fuzzy normal subgroup lattice* of G, respectively. Their binary operations $\wedge, \vee$ are defined by

$$\mu \wedge \eta = \mu \cap \eta,\ \mu \vee \eta = \langle \mu \cup \eta \rangle \quad \text{for all } \mu, \eta \in FL(G),$$

where $\langle \mu \cup \eta \rangle$ denotes the fuzzy subgroup of G generated by $\mu \cup \eta$ (that is, the intersection of all fuzzy subgroups of G containing both μ and η).

For each $\alpha \in [0, 1]$, we define the level subset

$$_{\mu}G_{\alpha} = \{x \in G | \mu(x) \geqslant \alpha\}.$$

These subsets allow us to characterize the fuzzy (normal) subgroups of G, in the following manner: μ is a fuzzy (normal) subgroup of G if and only if its level subsets are (normal) subgroups in G. We also mention that the fuzzy subgroup $\mu^* = \langle \mu \rangle$ generated by $\mu \in \mathscr{F}(G)$ can be described by using the level subsets $_{\mu}G_{\alpha}$, $\alpha \in [0, 1]$:

$$\mu^*(x) = \sup\{\alpha | x \in \langle _{\mu}G_{\alpha} \rangle\} \quad \text{for any } x \in G.$$

The fuzzy subgroups of G can be classified up to some natural equivalence relations on $\mathscr{F}(G)$. One of them (used in [25,26], too) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y) \quad \text{for all } x, y \in G)$$