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Nth-order fuzzy linear differential equations

T. Allahviranloo *, E. Ahmady, N. Ahmady

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

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Abstract

In this paper a numerical method for solving nth-order linear differential equations with fuzzy initial conditions is considered. The idea is based on the collocation method. The existence theorem of the fuzzy solution is considered. This method is illustrated by solving several examples.

Keywords: Fuzzy number; Extension principle; Basic functions; Fuzzy distance; Fuzzy differential equations

1. Introduction

The topic of fuzzy differential equations (FDEs) has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [10]; it was followed up by Dubois and Prade [13], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [24] and Goetschel and Voxman [14]. Kandel and Byatt [19,20] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [17,18], Seikkala [25], He and Yi [15], Kloeden [21] and Menda [23], and by other researchers (see [4,6,8,9,11,12,16]). The numerical methods for solving fuzzy differential equations are introduced in [1–3]. Buckley and Feuring [7] introduced two analytical methods for solving nth-order linear differential equations with fuzzy initial conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, in that they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function. In this paper, a numerical method to solve nth-order linear differential equations with fuzzy initial conditions is presented. The structure of the paper is organized as follows:

In Section 2, some basic definitions which will be used later in the paper are provided. In Section 3, one method for solving nth-order fuzzy differential equations is introduced, then the proposed method is illustrated by solving several examples in the same section, and the conclusion is drawn in Section 4.

* Corresponding author. Tel.: +989123508816.
E-mail address: tofigh@allahviranloo.com (T. Allahviranloo).

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2. Preliminaries

A tilde is placed over a symbol to denote a fuzzy set, as in $\tilde{\alpha}_1, \tilde{f}(t), \ldots$.

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \bar{u}(r)), 0 \leqslant r \leqslant 1$, which satisfy the following requirements.

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$.
- $\bar{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$.
- $\underline{u}(r) \leqslant \bar{u}(r), 0 \leqslant r \leqslant 1$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. This means that if $v \in E$ then the r-level set

$$[v]^r = \{s|v(s) \geqslant r\}, \quad 0 < r \leqslant 1$$

is a closed bounded interval which is denoted by

$$[v]^r = [\underline{v}(r), \bar{v}(r)].$$

For arbitrary $u = (\underline{u}, \bar{u}), v = (\underline{v}, \bar{v})$ and $k \geqslant 0$, addition and multiplication by k are defined as follows:

$$(\underline{u+v}) = \underline{u}(r) + \underline{v}(r),$$
$$(\overline{u+v}) = \bar{u}(r) + \bar{v}(r),$$
$$(\underline{ku})(r) = k\underline{u}(r), \quad (\overline{ku})(r) = k\bar{u}(r).$$

Definition 2.1. For arbitrary fuzzy quantities $u = (\underline{u}, \bar{u})$ and $v = (\underline{v}, \bar{v})$, the quantity

$$D(u,v) = \left[\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 \mathrm{d}r + \int_0^1 (\bar{u}(r) - \bar{v}(r))^2 \mathrm{d}r\right]^{\frac{1}{2}} \tag{2.1}$$

is the distance between u and v.

3. New solution

In this section, we are going to solve the following problem (taken from [7], Eq.1)

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t), \tag{3.2}$$

where $a_i(t)$, $0 \leqslant i \leqslant n-1$, are continuous on some interval I, subject to initial conditions

$$\tilde{y}(0) = \tilde{b}_0, \quad \tilde{y}'(0) = \tilde{b}_1, \ldots, \tilde{y}^{(n-1)}(0) = \tilde{b}_{n-1} \tag{3.3}$$

for fuzzy numbers $\tilde{b}_i$, $0 \leqslant i \leqslant n-1$. The interval I can be $[0,T]$ for some $T > 0$.

Buckly-Feuring method of solution is to fuzzify the crisp solution to obtain a fuzzy function $\overline{Y}(t)$, and then check to see if it satisfies the differential equation with fuzzy initial conditions. In this paper we proposed the other method for solving n-order fuzzy differential equation. This method is to seek an approximate solution as

$$\tilde{y}_N(t) = \sum_{k=0}^{N} \tilde{\alpha}_k \phi_k(t), \tag{3.4}$$

where $\phi_k(t)$ are positive basic functions whose all differentiations are positive. Now, the aim is to compute the fuzzy coefficients in (3.4) by setting the error to zero as follows,

$$\begin{aligned}\text{Error} = {} & D(\tilde{y}^{(n)} + a_{n-1}(t)\tilde{y}^{(n-1)} + \cdots + a_1(t)\tilde{y}' + a_0(t)\tilde{y}, \tilde{g}(t)) + D(\tilde{y}(0), \tilde{b}_0) + D(\tilde{y}'(0), \tilde{b}_1) + \cdots \\ & + D(\tilde{y}^{(n-1)}(0), \tilde{b}_{n-1}).\end{aligned} \tag{3.5}$$