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Initial value problem for fuzzy differential equations under dissipative conditions

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ABSTRACT

A new concept of inner product on the fuzzy space (E^n, D) is introduced, studied and used to prove several theorems stating the existence, uniqueness and boundedness of solutions of fuzzy differential equations. A stability result is also proved in the same context.

1. Introduction

The theory of fuzzy differential equations has attracted much attention in recent times because this theory represents a natural way to model dynamical systems under uncertainty. The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [4,18] and set-valued differential equations [16]. Using the Hukuhara derivative of multivalued functions, Puri and Ralescu [25] introduced the concept of *H*-differentiability for fuzzy functions. This concept has been studied and applied in context of fuzzy differential equations by Seikkala [27] and Kaleva [10] in time dependent form. Kaleva showed in [10,11] that, if f is continuous and satisfies the Lipschitz condition with respect x, then there exists a unique local solution for the fuzzy initial value problem $u'(t) = f(t, u)$, $u(0) = u_0$ on (E^n, D). Also, in [11], he has shown that the Peano theorem does not hold because the metric space (E^n, D) is not locally compact. Nieto [19] proved the existence theorem of Peano for fuzzy differential equations on (E^n, D) if f is continuous and bounded. Buckley and Feuring [3] gave a very general formulation of a fuzzy first-order initial value problem. This approach based on *H*-differentiability has the disadvantage that any solution of a fuzzy differential equation has increasing length of its support. Consequently, this approach cannot really reflect any of the rich behavior of ordinary differential equations such as periodicity, stability, bifurcation, and the like and is ill suited for modeling [5]. This shortcoming was resolved by interpreting a fuzzy differential equations as a family of differential inclusions [9]. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-number-valued function. Recently, Bede and Gal [2] solved the above mentioned approach under strongly generalized differentiability of fuzzy-number-valued functions In this case the derivative exists and the solution of a fuzzy differential equation may have decreasing length of the support, but the uniqueness is lost. The existence theorems of solutions for fuzzy initial value problem under different sets of assumptions are given in

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[1,6–8,12,17,20,21,23,28,29,31,33]. For a multitude of other concepts and results on fuzzy differential equations we refer to the monograph [14].

We consider the fuzzy initial value problem

$$u' = f(t,u), \quad u(t_0) = u_0 \in E^n, \ \ t_0 \geqslant 0, \tag{1.1}$$

where $f : [0,\infty) \times E^n \to E^n$. In the paper [22], Park and Han investigated the existence and uniqueness of fuzzy initial value problem (1.1) under some dissipative conditions via the classical Rådström embedding theorem [26]. Also, in the paper [30], Song et al. investigated the same problem under Lyapunov dissipative-type conditions.

In this paper, we introduce a new concept of inner product on the fuzzy space (E^n, D). By help of this inner product we formulate some dissipative conditions for Cauchy problem (1.1) and, under these conditions, we prove several theorems stating the existence, uniqueness and boundedness of solutions of fuzzy differential equations. Also, in the same context, we prove a stability result for trivial solution of (1.1).

2. Preliminaries

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact convex subsets of $\mathbb{R}^n$. We define the Hausdorff distance between sets $A, B \in K_c(\mathbb{R}^n)$ by

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\}.$$

Denote

$$E^n = \{u : \mathbb{R}^n \to [0,1]; u \text{ satisfies (i)–(iv) below}\}.$$

(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leqslant \lambda \leqslant 1$,

$$u(\lambda x + (1-\lambda)y) \geqslant \min\{u(x), u(y)\};$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = cl\{x \in \mathbb{R}^n; u(x) > 0\}$ is compact.

For $0 < \alpha \leqslant 1$, denote $[u]^\alpha = \{x \in \mathbb{R}^n; u(x) \geqslant \alpha\}$. Then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in K_c(\mathbb{R}^n)$ for all $0 < \alpha \leqslant 1$. For later purposes, we define $\hat{0} \in E^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.

If we define

$$D[u,v] = \sup_{0\leqslant\alpha\leqslant 1} d_H([u]^\alpha, [v]^\alpha),$$

then it is well known that D is a metric in E^n and that (E^n, D) is a complete metric space [10,25].

We list the following properties of $D[u,v]$:

$$D[u+w, v+w] = D[u,v] \quad \text{and} \quad D[u,v] = D[v,u],$$
$$D[\lambda u, \lambda v] = |\lambda| D[u,v],$$
$$D[u,v] \leqslant D[u,w] + D[w,v]$$

for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$.

In the following, we recall some main concepts and properties of differentiability and integrability for fuzzy functions [10,24,25].

If there exists $w \in E^n$ such that $u = v + w$, then w is called the H-difference of u and v and is denoted by $u - v$.

Let I be an interval in $\mathbb{R}$. A mapping $F : I \to E^n$ is differentiable at $t_0 \in I$ if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h}$$

exist and equal to $F'(t_0)$.

The fuzzy valued function $F : I \to E^n$ is called strongly measurable, if for each $\alpha \in [0,1]$ the set-valued function $F_\alpha : I \to K_c(\mathbb{R}^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha$$

is Lebesgue measurable. A function $F : I \to E^n$ is called integrable bounded, if there exists an integrable function $h : I \to \mathbb{R}_+$ such that $D[F_0(t), \hat{0}] \leqslant h(t)$, for all $t \in I$. Let $F : I \to E^n$. The integral of F over I, denoted $\int_I F(t)\mathrm{d}t$ is defined levelwise by the equation

$$\left[\int_I F(t)\mathrm{d}t\right]^\alpha = \int_I F_\alpha(t)\mathrm{d}t = \left\{\int_I f(t)\mathrm{d}t; f : I \to \mathbb{R}^n \text{ is a measurable selection for } F_\alpha\right\}$$

for all $0 \leqslant \alpha \leqslant 1$.