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The structure of decomposable indistinguishability operators

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ABSTRACT

Decomposable fuzzy relations are studied.
Symmetric fuzzy relations are proved to be generated by a single fuzzy subset.
For Archimedean t-norms, decomposable indistinguishability operators generate special kinds of betweenness relations that characterize them.
A new way to generate indistinguishability operators coherent with the underlying ordering structure of the real line is given in the sense that this structure is compatible with the betweenness relation generated by the relation is developed.

1. Introduction

Indistinguishability operators [3,8,18], also known as fuzzy equalities and fuzzy equivalence relations, are one of the most important types of fuzzy relation since they fuzzify the concepts of crisp equality and equivalence relation.

One of the most interesting issues related to them is their generation. This depends on the way the data are given and of the use we want to make of them [5,16,17]. The three most common ways are calculating the T-transitive closure of a reflexive and symmetric fuzzy relation, using the Representation Theorem and calculating a decomposable operator from a pair of fuzzy subsets [7,9,13,15].

Decomposable fuzzy relations have been applied successfully in Mamdani controllers, first using the Minimum t-norm and then with more general t-norms. This paper is focused on decomposable indistinguishability operators when the t-norm is continuous Archimedean or the Minimum.

After a section of preliminary results, Section 3 is devoted to the study of symmetric fuzzy relations. The main result is that they can be generated by a single fuzzy subset.

Section 4 deals with decomposable indistinguishability operators. By their metric nature [10], decomposable indistinguishability operators with respect to an Archimedean t-norm generate a special kind of betweenness relations that characterize them.

Section 5 studies a new kind of indistinguishability operators generated by fuzzy numbers that respect the ordering of the real line: In the betweenness relation they generate, if $x < y < z$, then y lies between x and z.

Section 6 ends this paper.

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2. Preliminaries

This section contains the definitions of decomposable fuzzy relation, indistinguishability operator, quasi-arithmetic means and some of their properties that will be used throughout the paper.

Definition 2.1. For a given t-norm T, a fuzzy relation R on a set X is T-decomposable if and only if there exists a couple of fuzzy subsets μ, v of X such that for all $x, y \in X$

$$R(x,y) = T(\mu(x), v(y)).$$

We will say that the pair (μ, v) generates R. If $\mu = v$, then we will simply say that μ generates R.

Definition 2.2 [18]. Given a t-norm T, a fuzzy relation E on a set X is a T-indistinguishability operator on X if and only if for all $x, y, z \in X$ satisfies

1. $E(x,x) = 1$ (Reflexivity)
2. $E(x,y) = E(y,x)$ (Symmetry)
3. $T(E(x,y), E(y,z) \leqslant E(x,z)$ (T-transitivity).

E separates points if and only if $E(x,y) = 1$ implies $x = y$.

$E(x,y)$ can be interpreted as the degree of indistinguishability, equality or equivalence between x and y. For a survey on T-indistinguishability operators, readers can have a look at [3].

Definition 2.3. Given a continuous t-norm T, its residuation $\vec{T}$ is defined for all $x, y \in [0,1]$ by

$$\vec{T}(x|y) = \sup\{z \in [0,1] | T(z,x) \leqslant y\}.$$

Definition 2.4. Given a continuous t-norm T, its biresiduation $\overleftrightarrow{T}$ is defined for all $x, y \in [0,1]$ by

$$\overleftrightarrow{T}(x,y) = T(\vec{T}(x|y), \vec{T}(y|x)) = \mathrm{Min}(\vec{T}(x|y), \vec{T}(y|x)).$$

Proposition 2.5. *Let T be a continuous t-norm and μ a fuzzy subset of a set X. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x,y) = \overleftrightarrow{T}(\mu(x), \mu(y))$$

is a T-indistinguishability operator.

T-indistinguishability operators generated by a fuzzy subset as in the previous proposition are called one-dimensional.

Theorem 2.6 (Representation Theorem [14]). *Let R be a fuzzy relation on a set X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(\mu_i)_{i \in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = inf_{i \in I} E_{\mu_i}(x,y).$$

Definition 2.7. Let E be a T-indistinguishability operator on X. The dimension of E is the minimum of the cardinalities of the generating families of E in the sense of the Representation Theorem. A generating family with this cardinality is called a basis of E.

In Section 4, a characterization of one-dimensional T-indistinguishability operators for continuous Archimedean t-norms will be provided.

The next proposition characterizes continuous Archimedean t-norms.

Proposition 2.8. *A continuous t-norm T is Archimedean if and only if there exists a continuous decreasing map $t : [0,1] \to [0, \infty]$ with $t(1) = 0$ such that for all $x, y \in [0,1]$*

$$T(x,y) = t^{[-1]}(t(x) + t(y)),$$

where

$$t^{[-1]} = \begin{cases} t^{-1}(x) & \text{if } t(x) \leqslant t(0), \\ 0 & \text{otherwise}. \end{cases}$$

t is called an additive generator of T.

Theorem 2.9. *Let E be a Min-indistinguishability operator on a set X separating points. E is one-dimensional if and only if there exists $a \in X$ such that the fuzzy subset $E(a, \bullet)$ is a one to one map.*