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Level sets and the extension principle for interval valued fuzzy sets and its application to uncertainty measures

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ABSTRACT

We describe the representation of a fuzzy subset in terms of its crisp level sets. We then generalize these level sets to the case of interval valued fuzzy sets and provide for a representation of an interval valued fuzzy set in terms of crisp level sets. We note that in this representation while the level sets are crisp the memberships are still intervals. Once having this representation we turn to its role in the extension principle and particularly to the extension of measures of uncertainty of interval valued fuzzy sets. Two types of extension of uncertainty measures are investigated. The first, based on the level set representation, leads to extensions whose values for the measure of uncertainty are themselves fuzzy sets. The second, based on the use of integrals, results in extensions whose value for the uncertainty of an interval valued fuzzy sets is an interval.

1. Introduction

Within the framework of fuzzy set theory Zadeh's extension principle [16,19,21,11] provides a crucial tool for enabling the performance of mathematical operations and algorithms when data values are expressed as fuzzy sets. In the case when the operation to be extended is an operation defined via a mapping whose arguments are crisp subsets, such as probability measures, the representation theorem [18] must often be used to provide a crisp set representation of a fuzzy set in terms of level sets. Our objective here is to provide the basis for the extension of set mapping operations to the case of interval valued fuzzy sets. To accomplish this we need to introduce the idea of the level sets of interval fuzzy sets and the related formulation of a representation of an interval valued fuzzy set in terms of its level sets. Once having these structures we then can provide the desired extension to interval valued fuzzy sets. Here we apply our tools to the extension of measures of uncertainty of interval valued fuzzy sets.

2. Level sets and the representation theorem

Assume F is a standard fuzzy subset of a finite set X. We recall that the level sets associated with F are defined as $F_\alpha = \{x|F(x) \geqslant \alpha\}$. The level sets are crisp subsets of X. They are the subsets of X with membership grade at least α.

The representation theorem [18] provides a method for expressing F in terms of the level sets in particular $F = \cup_{\alpha \in [0,1]} \alpha F_\alpha$ where αF_α is a fuzzy subset of X with membership function $\alpha F_\alpha(x) = \alpha$ if $x \in F_\alpha$ and $\alpha F_\alpha(x) = 0$ if $x \notin F_\alpha$. If we let I^α be a fuzzy subset such that for all x we have $I^\alpha(x) = \alpha$ we see that $\alpha F_\alpha = F_\alpha \cap I^\alpha$. From this we see that $\alpha F_\alpha(x) = F_\alpha(x) \wedge \alpha = \text{Min}[F_\alpha(x), \alpha]$. We also see that $\alpha F_\alpha(x)$ is the product of α and $F_\alpha(x)$.

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We note that for $F = \cup_{\alpha\in[0,1]}\alpha F_\alpha$ then $F(x) = \text{Max}_\alpha[\alpha F_\alpha(x)]$. It is the largest membership of x in any of the fuzzy subsets αF_α.

In the case where X is finite there can be only a finite set of distinct membership grades of the elements of X in F. If we denote these as β_j and index them in increasing order

$$\beta_0 < \beta_1 < \beta_2 < \beta_3 \cdots < \beta_m,$$

where we let $\beta_0 = 0$. In this case there are $m + 1$ distinct level sets associated with F. In particular

$$F_\alpha = \{x|F(x) \geqslant \beta_j\}, \quad \beta_{j-1} < \alpha \leqslant \beta_j \quad \text{for } j = 1 \text{ to } m.$$

We now suggest an alternative methodology for obtaining the representation theorem. Again let F be a fuzzy subset of the finite set X. Let I^X be the power set of X, it is the set of all subsets of X. For any $B \in I^X$ we define

$$\text{Val}_F(B) = \text{Min}_{x\in B}[F(x)].$$

Thus $\text{Val}_F(B)$ is the min membership grade of the elements of B in F. $\text{Val}_F(B)$ can also be expressed as $\text{Val}_F(B) = \text{Min}_x[(F(x) \wedge B(x)) \vee \overline{B}(x)]$.

Let us denote $D = \cup_{B\subseteq X}\{\text{Val}_F(B)\}$, it is the set of all distinct values of $\text{Val}_F(B)$. For any $\alpha \in D$ we let $P_\alpha = \{B|\text{Val}_F(B) = \alpha\}$, it is the collection of all subsets of X whose value is α. P_α is a subset of the power set of X. Let us now denote $B_\alpha = \bigcup_{B\in P_\alpha} B$, it is the union of all the subsets that have $\text{Val}_F(B) = \alpha$. Using this we can obtain a form of the representation theorem. In particular

$$F = \bigcup_{\alpha\in D} \alpha B_\alpha,$$

where again αB_α is defined as a fuzzy subset whose membership grade $\alpha B_\alpha(x) = \alpha \wedge B_\alpha(x)$. It is worth emphasizing the range of α in the preceding union is only the subset D, the set of distinct membership grades associated with F.

3. A representation theorem for interval valued fuzzy sets

Recent interest has focused on type 2 fuzzy subsets, particularly interval valued type 2 [5,9]. Again let X be a finite set. An interval valued fuzzy subset (IVFS) A is one in which the membership grades are intervals, $A(x) = [L(x), R(x)]$. Here we require that $L(x)$ and $R(x) \in [0,1]$ and $R(x) \geqslant L(x)$. We note that if $L(x) = R(x)$ then the membership grade $A(x) = L(x) = R(x)$.

The manipulation of these interval valued fuzzy subsets makes considerable use of interval arithmetic. We shall briefly describe some of the operators. Assume A_1 and A_2 are two IVFS, their intersection $D = A_1 \cap A_2$ is also an IVFS such $D(x) = \text{Min}(A_1(x), A_2(x))$. We note that the minimum of two intervals is defined as

$$\text{Min}(A_1(x), A_2(x)) = [\text{Min}(L_1(x), L_2(x)), \text{Min}(R_1(x), R_2(x))],$$

hence we get $D(x) = [L_1(x) \wedge L_2(x), R_1(x) \wedge R_2(x)]$.

More generally if $A_1, \ldots, A_q$ are all IVFS then $F = A_1 \cap A_2 \cdots \cap A_q$ is an IVFS with membership grade $F(x) = \text{Min}_i(A_i(x)) = [\text{Min}_i L_i(x), \text{Min}_i R_i(x)]$.

The union of IVFS is defined in a similar manner with Max replacing Min. Thus if $E = A_1 \cup A_2 \cdots \cup A_q$ then E is an IVFS where $E(x) = [\text{Max}_i L_i(x), \text{Max}_i R_i(x)]$. The negation of an IVFS is denoted $\overline{A}$ and if $A(x) = [L(x), R(x)]$ then $\overline{A}(x) = [1 - R(x), 1 - L(x)]$.

We look at these operations for some special cases of membership grade. Two important membership grades are $A(x) = 0$ and $A(y) = 1$ which are represented in an IVFS respectively as $A(x) = [0,0]$ and $A(y) = [1,1]$. We easily see that

$$[L,R] \wedge [0,0] = [0,0],$$
$$[L,R] \wedge [1,1] = [L,R]$$

and

$$[L,R] \vee [0,0] = [L,R],$$
$$[L,R] \vee [1,1] = [1,1].$$

Another notable membership grade in the case of IVFS's is unknown, here $A(x) = [0,1]$. We see that in the case of this membership grade we get

$$[L,R] \wedge [0,1] = [0,R],$$
$$[L,R] \vee [0,1] = [L,1].$$

We note that if $A(x) = [0,1]$ then $\overline{A}(x) = [0,1]$.

We now turn to the task of providing a representation theorem for interval valued fuzzy sets. We shall provide an alternative method of getting the representation theorem introduced earlier.

Let A be an IVFS on the finite set X. We denote its membership grades as $A(x) = [L(x), R(x)]$. Again let I^X be the power set of X. For any $B \in I^X$ we define

$$\text{Val}_A(B) = \text{Min}_{x\in B}[A(x)].$$