Available online at www.sciencedirect.com
ScienceDirect
Information Sciences 177 (2007) 2336–2348
INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL
www.elsevier.com/locate/ins

Distance and similarity measures for fuzzy operators

Victor Balopoulos, Anestis G. Hatzimichailidis, Basil K. Papadopoulos *

Department of Civil Engineering, Democritus University of Thrace, GR-67100 Xanthi, Greece

Received 7 February 2006; received in revised form 5 January 2007; accepted 8 January 2007

Abstract

We introduce and study a new family of normalized distance measures between binary fuzzy operators, along with its dual family of similarity measures. Both are based on matrix norms and arise from the study of the aggregate plausibility of set-operations. We also suggest a new family of normalized distance measures between fuzzy sets, based on binary operators and matrix norms, and discuss its qualitative and quantitative features. All measures proposed are intended for applications and may be customized according to the needs and intuition of the user.

Keywords: Fuzzy operator; Fuzzy implication; Metric distance; Similarity measure; Fuzzy sets

1. Introduction and preliminaries

In this section, we review standard notations and definitions from the literature of fuzzy sets [5,6,8,12,14,17], regarding operators (implication, product, and aggregation) and metrics for fuzzy sets. In Section 2, we propose and study a family of normalized distance measures for binary fuzzy operators in general (and fuzzy implications in particular), its dual family of similarity measures, and an analogously motivated family of distance measures for fuzzy sets. Section 3 contains numerical examples and, finally, Section 4, recapitulates the results presented in this paper and comments on the usefulness of the measures proposed.

For any finite universe set $X = \{x_1, \dots, x_n\}$ the set of fuzzy sets on it is denoted by $FS(X)$. For any two fuzzy sets $A, B \in FS(X)$, with membership functions μ and v, respectively, various distance measures have been proposed [2,4,11]. The ones most commonly employed in practice are

* Corresponding author. Tel.: +30 541 27408.
E-mail address: papadob@civil.duth.gr (B.K. Papadopoulos).

0020-0255/$ - see front matter
doi:10.1016/j.ins.2007.01.005

The *Hamming* distance $d_{\mathrm{H}}(\mu, v) = \sum_{i=1}^{n} |\mu(x_i) - v(x_i)|$,

The *normalized Hamming* distance $d_{\mathrm{nH}}(\mu, v) = \frac{1}{n}\sum_{i=1}^{n} |\mu(x_i) - v(x_i)|$,

The *Euclidean* distance $d_{\mathrm{E}}(\mu, v) = \sqrt{\sum_{i=1}^{n} (\mu(x_i) - v(x_i))^2}$,

The *normalized Euclidean* distance $d_{\mathrm{nE}}(\mu, v) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\mu(x_i) - v(x_i))^2}$, and

The *maximum* distance $d_{\infty}(\mu, v) = \max_i |\mu(x_i) - v(x_i)|$.

Implication is a function that maps the truth values of any pair of propositions p and q to the truth value of the conditional proposition "if p then q". In classical set theory, implication is the mapping $m : \{0,1\} \times \{0,1\} \to \{0,1\}$ with $m(1,0) = 0$ and $m(0,0) = m(0,1) = m(1,1) = 1$. Similarly, in fuzzy set theory, *a fuzzy implication*, $\sigma(a,b) \in \Im$, is any extension of the classical implication to fuzzy propositions p with plausibility $a \in [0,1]$ and q with plausibility $b \in [0,1]$, which at least satisfies the boundary conditions $\sigma(1,0) = 0$ and $\sigma(0,0) = \sigma(0,1) = \sigma(1,1) = 1$ [12].

The potential variety of fuzzy implications is obviously infinite, but those employed in practice belong to at least one of three general classes (see [6,8,12,14]):

(i) S-implications $\sigma(a,b) = S(N(a), b)$,
(ii) R-implications $\sigma(a,b) = \sup\{x \in [0,1] \mid T(a,x) \leqslant b\}$, and
(iii) QL-implications $\sigma(a,b) = S(N(a), T(a,b))$, for some operator triplet $\langle T, S, N\rangle$ on $[0,1]$ consisting of a t-norm T, a t-conorm S, and a fuzzy negation N, which together obey the De Morgan laws.

Aggregation or *averaging* operations (see, e.g., [12]) allow multiple fuzzy sets to be combined. They are realized by functions of the form:

$$H : [0,1]^n \to [0,1], \quad n \geqslant 2,$$

that are *continuous* and *increasing* in all arguments, and satisfy the classical boundary conditions $H(0,\ldots,0) = 0$ and $H(1,\ldots,1) = 1$. Where all sets are of equal importance, aggregation operators are also required to be *symmetric* in all arguments (unaffected by permutations) and *idempotent* (i.e., to satisfy $H(a,\ldots,a) = a, \forall a \in [0,1]$). The most commonly used family of such operators are the *generalized means* defined by

$$H_{\beta}(a_1,\ldots,a_n) \triangleq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\beta}\right)^{1/\beta}.$$

Generalized means include the maximum, the root-mean-square, the arithmetic mean, the geometric mean, the harmonic mean, and the minimum for $\beta = \infty, 2, 1, 0, -1$, and $-\infty$, respectively. Since some arguments of H_{β} may be zero, it is only generalized means with $\beta > 0$ (usually with $\beta \in \{1,2\}$) that are of practical importance as aggregation operators.

Contrary to the above *theoretical* definitions, in fuzzy systems *applications* the predominant practice (known as the *Mamdani* method [12]) is to replace fuzzy implications with fuzzy *products* (symmetric operators, formally akin to t-norms), and to aggregate the results by union (usually the $\max\{\cdots\}$). The fuzzy products most commonly used in applications are

The *Mamdani* rule $\sigma_{\mathrm{M}}(\mu(x_i), v(y_j)) = \min\{\mu(x_i), v(y_j)\}$ and

The *Larsen* rule $\sigma_{\mathrm{La}}(\mu(x_i), v(y_j)) = \mu(x_i) * v(y_j)$.

Fuzzy products clearly do not reduce to the classical implication in the limit. In fact, they differ fundamentally from fuzzy implications in that they: