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# The semiring of matrices over a finite chain ☆

Xianzhong Zhao [a,*,1], Young Bae Jun [b], Fang Ren [c]

[a] Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330027, PR China
[b] Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Republic of Korea
[c] Department of Mathematics, Northwest University, Xi'an, Shaanxi 710069, PR China

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ABSTRACT

Let $L_m$ denote the chain $\{0,1,2,\ldots,m-1\}$ with the usual ordering and $M_n(L_m)$ the matrix semiring of all $n\times n$ matrices with elements in $L_m$. We firstly introduce some order-preserving semiring homomorphisms from $M_n(L_m)$ to $M(L_k)$. By using these homomorphisms, we show that a matrix over the finite chain $L_m$ can be decomposed into the sum of some matrices over the finite chain $L_k$, where $k<m$. As a result, cut matrices decomposition theorem of a fuzzy matrix (Theorem 4 in [Z.T. Fan, Q.S. Cheng, A survey on the powers of fuzzy matrices and FBAMs, International Journal of Computational Cognition 2 (2004) 1–25 (invited paper)]) is generalized and extended. Further, we study the index and periodicity of a matrix over a finite chain and get some new results. On the other hand, we introduce a semiring embedding mapping from the semiring $M_n(L_m)$ to the direct product of the $h$ copies of the semiring $M_n(L_k)$ and discuss Green's relations on the multiplicative semigroup of the semiring $M_n(L_m)$. We think that some results obtained in this paper is useful for the study of fuzzy matrices.



## 1. Introduction and preliminaries

A *semiring* is an algebra $(S,+,\cdot)$ with two binary operations + and $\cdot$ such that both the reducts $(S,+)$ and $(S,\cdot)$ are semigroups and such that the distributive laws $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$ hold [1–4]. A semiring $S$ is said to be a partially ordered semiring if it admits a compatible ordering $\leqslant$; that is, $\leqslant$ is a partial order on $S$ satisfying the following condition: for any $a,b,c,d\in S$,

$$\text{if } a\leqslant b \text{ and } c\leqslant d, \text{ then } a+c\leqslant b+d \text{ and } ac\leqslant bd.$$

A partially ordered semiring $S$ is said to be a totally ordered semiring if the imposed partial order is a total order [6,7,15,16].

Throughout this paper, we always assume that $m$ and $k$ are any (but fixed) two positive integers with $1<k<m$ and that $h=\lceil\frac{m-1}{k-1}\rceil$ is the least integer greater than or equal to $\frac{m-1}{k-1}$.

Let $L_m$ be the chain $\{0,1,2,\ldots,m-1\}$ with the usual ordering. Define the addition and the multiplication on $L_m$ as follows:

$$(\forall a,b\in L_m)\, a+b=\max\{a,b\} \quad \text{and} \quad ab=\min\{a,b\}.$$

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* Corresponding author. Tel.: +86 0791 812036, fax: +86 0791 8120368.
*E-mail address:* xianzhongzhao@263.net (X. Zhao).

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Then it is easily seen that $(L_m, +, \cdot, \leqslant)$ is a totally ordered semiring. Let $M_n(L_m)$ denote the set of all $n \times n$ matrices with elements in $L_m$. As usual, we define the addition and the multiplication on $M_n(L_m)$ as follows:

$$(\forall A = (a_{ij}), B = (b_{ij}) \in M_n(L_m)) A + B = C \quad \text{and} \quad AB = D,$$

where $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \max\{\min\{a_{i1}, b_{1j}\}, \min\{a_{i2}, b_{2j}\}, \ldots, \min\{a_{in}, b_{nj}\}\}$ and $c_{ij} = a_{ij} + b_{ij} = \max\{a_{ij}, b_{ij}\}$. Then $(M_n(L_m), +, \cdot)$ is a semiring. Equip this semiring $(M_n(L_m), +, \cdot)$ with the partial order relation

$$A \leqslant B \Longleftrightarrow a_{ij} \leqslant b_{ij}.$$

Then $(M_n(L_m), +, \cdot, \leqslant)$ is a semilattic ordered semiring. That is, $(M_n(L_m), \leqslant)$ is a semilattice and $(M_n(L_m), +, \cdot, \leqslant)$ is a partially ordered semiring.

By replacing $L_m$ with the closed interval $[0, 1]$ (a infinite chain with the usual ordering) in the above definitions, we may get so-called $n \times n$ fuzzy matrices and fuzzy matrix semirings. The fuzzy matrices have been extensively studied [1–5,8–14]. We notice that the some problems in theory of fuzzy matrices may be translated into the corresponding problems in the theory of semilattice-ordered semirings $M_n(L_m)$. For instance, given any (but fixed) $n \times n$ fuzzy matrix $A = (a_{ij})$, without loss of generality, suppose that $a_0 < a_1 < a_2 < \cdots < a_{m-1}$ is the sequence of all the distinct elements of $A$ and that $B = (b_{ij})$, where $b_{ij} = t$ if $a_{ij} = a_t$. It is easily seen that $B \in M_n(L_m)$ and that the index and period of fuzzy matrix $A$ are equal to the index and period of matrix $B$, respectively. Thus, to discuss the index and period of a fuzzy matrix $A$, we may only study the index and period of matrix $B$. This also shows that our studies on the $n \times n$ matrices over the finite chain $L_m$ and on the matrix semiring $M_n(L_m)$ are useful for the study of fuzzy matrices.

In Section 2, some order-preserving semiring homomorphisms from $M_n(L_m)$ to $M(L_k)$ are introduced and studied. By using these homomorphisms, it is shown in Section 3 that a matrix over the finite chain $L_m$ can be decomposed into the sum of some matrices over the finite chain $L_k$, where $k < m$. As a result, cut matrices decomposition theorem of a fuzzy matrix (Theorem 4 in [1]) is generalized and extended. Further, the index and periodicity of a matrix over a finite chain are studied and some new results are obtained. On the other hand, a semiring embedding mapping from the semiring $M_n(L_m)$ to the direct product of the $h$ copies of the semiring $M_n(L_k)$ is introduced and the Green's relations for the multiplicative semigroup of the semiring $M_n(L_m)$ are discussed in Section 4.

## 2. Order-preserving semiring homomorphisms

**Definition 1.** For any $1 \leqslant s \leqslant h$, define the mapping $\varphi_{mk}^{(s)}$ from $L_m$ to $L_k$ by

$$\varphi_{mk}^{(s)}(i) = \begin{cases} 0, & i < (k-1)(s-1), \\ j, & (k-1)(s-1) \leqslant i = (k-1)(s-1) + j \leqslant (k-1)s, \\ k-1, & i > (k-1)s. \end{cases}$$

**Example 1**

$$\varphi_{10,3}^{(1)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix};$$
$$\varphi_{10,3}^{(2)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix};$$
$$\varphi_{10,3}^{(3)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \end{pmatrix};$$
$$\varphi_{10,3}^{(4)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{pmatrix};$$
$$\varphi_{10,3}^{(5)} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the image of an element $i$ in first row stands under $i$ in each of the above matrices.

**Lemma 1.** *For any $i, j \in L_m$ with $i \leqslant j$ and any positive integers $s, t$ with $s \leqslant t \leqslant h$, we have*

$$\varphi_{mk}^{(s)}(i) \leqslant \varphi_{mk}^{(s)}(j); \tag{1}$$
$$\varphi_{mk}^{(t)}(i) \leqslant \varphi_{mk}^{(s)}(i). \tag{2}$$
$$\varphi_{mk}^{(s)}(i + j) = \varphi_{mk}^{(s)}(i) + \varphi_{mk}^{(s)}(j); \tag{3}$$
$$\varphi_{mk}^{(s)}(ij) = \varphi_{mk}^{(s)}(i)\varphi_{mk}^{(s)}(j). \tag{4}$$

**Proof.** For any $i, j \in L_m$ with $i \leqslant j$, by Definition 1, it is easily verified that $\varphi_{mk}^{(s)}(i) \leqslant \varphi_{mk}^{(s)}(j)$ holds in $L_k$ and $\varphi_{mk}^{(t)}(i) \leqslant \varphi_{mk}^{(s)}(i)$ for any $1 \leqslant s \leqslant t \leqslant h$. That is to say, (1) and (2) are true.