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# Diameter variability of cycles and tori

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ABSTRACT

The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. Graham and Harary [N. Graham, F. Harary, Changing and unchanging the diameter of a hypercube, Discrete Applied Mathematics 37/38 (1992) 265–274] studied how the diameter of hypercubes can be affected by increasing and decreasing edges. They concerned whether the diameter is changed or remains unchanged when the edges are increased or decreased. In this paper, we modify three measures proposed in Graham and Harary (1992) to include the extent of the change of the diameter. Let $D^{-k}(G)$ is the least number of edges whose addition to $G$ decreases the diameter by (at least) $k$, $D^{+0}(G)$ is the maximum number of edges whose deletion from $G$ does not change the diameter, and $D^{+k}(G)$ is the least number of edges whose deletion from $G$ increases the diameter by (at least) $k$. In this paper, we find the values of $D^{-k}(C_m)$, $D^{-1}(T_{m,n})$, $D^{-2}(T_{m,n})$, $D^{+1}(T_{m,n})$, and a lower bound for $D^{+0}(T_{m,n})$ where $C_m$ be a cycle with $m$ vertices, $T_{m,n}$ be a torus of size $m$ by $n$.



## 1. Introduction

### 1.1. Basis

Let $G = (V, E)$ be a graph. $V(G)$ is the vertex set of $G$ and $E(G) \subseteq V(G) \times V(G)$ is the edge set of $G$. Let $u$ and $v$ be two different vertices in a graph $G$. We say that $u$ and $v$ are *adjacent* if $(u, v) \in E(G)$. A path from $u$ to $v$, delimited by $\langle u = x_0, x_1, x_2, \ldots, x_k = v \rangle$, is a sequence of distinct vertices such that $x_i$ and $x_{i+1}$ are adjacent for $0 \leqslant i \leqslant k-1$. The length of a path is the number of edges in it. The *distance* between $u$ and $v$ in $G$, denoted as $d_G(u, v)$, is the length of a shortest path joining them. The *diameter* of a graph $G$, denoted as $D(G)$, is the maximum distance between any two vertices.

An interconnection network connects the processors of a parallel and distributed system. The topology of an interconnection network for a parallel and distributed system can always be represented by a graph, where each vertex represents a processor and each edge represents a vertex-to-vertex communication link. Communication is a critical issue in the design of a parallel and distributed system. The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. To expedite communication, the smaller diameter is preferred. Besides, in order to increase the transmission rate and enhance the transmission reliability, it is also important to construct vertex-disjoint paths between any two vertices in a network [8,9].

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### 1.2. Definitions and properties

In fact, the diameter of a graph can be affected by adding or deleting edges [1,4,5]. For example, let $m$-cycle $C_m$ be a graph with vertex set $\{0,1,2,\dots,m-1\}$ and edge set $\{(i,i+1) \mid 0 \leqslant i \leqslant m-1\}$, where addition is in integer modulo $m$. It is known that $D(C_m) = \lfloor m/2 \rfloor$ [10]. Let $P_m$ be a graph with vertex set $\{0,1,2,\dots,m-1\}$ and edge set $\{(i,i+1) \mid 0 \leqslant i \leqslant m-2\}$, it is known that $D(P_m) = m-1$ [10]. It is easy to check that deleting any edge renders $C_m$ to a path of $m$ vertices and then the diameter increases to $m-1$. On the other hand, a cycle $C_m$ can be obtained by adding the edge $(0,m)$ to path $P_m$. The diameter decrease to $\lfloor m/2 \rfloor$.

Let $k$ be an arbitrary positive integer. The diameter variability arising from change of edges of graph $G$ is defined as follows:

$D^{-k}(G)$: the least number of edges whose addition to $G$ decreases the diameter by (at least) $k$;
$D^{+0}(G)$: the maximum number of edges whose deletion from $G$ does not change the diameter;
$D^{+k}(G)$: the least number of edges whose deletion from $G$ increases the diameter by (at least) $k$.

For example, $D^{-1}(P_m) = D^{-2}(P_m) = \cdots = D^{-(m-1-\lfloor m/2 \rfloor)}(P_m) = 1$ and $D^{+1}(C_m) = D^{+2}(C_m) = \cdots = D^{+(m-1-\lfloor m/2 \rfloor)}(C_m) = 1$. The $n$-dimensional hypercube, $Q_n$, consists of all $n$-bit binary strings as its vertices. Two vertices are adjacent if they differ only in one bit position. Graham and Harary [4] considered changing the diameter without considering the extent of the change, i.e., they considered $D^{-1}(G)$ and $D^{+1}(G)$. They showed that $D^{-1}(Q_n) = 2$, $D^{+1}(Q_n) = n-1$ and $D^{+0}(Q_n) \geqslant (n-3)2^{n-1}+2$. Bouabdallah et al. [1] improved the lower bound of $D^{+0}(Q_n)$ and furthermore gave an upper bound, $(n-2)2^{n-1} - \binom{n}{\lfloor n/2 \rfloor} + 2 \leqslant D^{+0}(Q_n) \leqslant (n-2)2^{n-1} - \lceil (2^n-1)/(2n-1) \rceil + 1$.

The *edge connectivity* of a graph $G$, denoted by $\kappa\prime(G)$, is the least number of edges whose deletion disconnects $G$. Clearly, $D^{+i}(G) \leqslant \kappa\prime(G)$ for all $i$. The diameter of a complete graph equals one. Given a graph $G$, for $1 \leqslant i \leqslant D(G)-1$, $D^{-i}(G)$ is no more than the number of edges needed to be added to $G$ to make $G$ be a complete graph. Note that $0 \leqslant D^{-i}(G) \leqslant D^{-j}(G)$ and $0 \leqslant D^{+i}(G) \leqslant D^{+j}(G)$ if $i \leqslant j$. For convenience, we write $D^{-1}(G)$ and $D^{+1}(G)$ as $D^{-}(G)$ and $D^{+}(G)$, respectively, throughout the paper.

In this paper, we study the change of diameter arising from the change of edges in cycles and tori. A cycle is the topological structure of a ring network. It is one of the most common, simple and useful interconnection networks [10]. More properties, performances, and details about cycles can be found in [2,7,10]. A torus, denoted as $T_{m,n}$, is a graph obtained by the Cartesian product of cycles $C_m$ and $C_n$. It is a 2-dimension array with wraparound wires in the rows and columns. The number of edges of $T_{m,n}$ is $2mn$ and it is known that the diameter of $T_{m,n}$ is $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$ [7]. For more details on properties and performances, such as throughput, latency, and path diversity, see [2].

## 2. Changing the diameter of cycles

Since deleting any edge renders $C_m$ to a path $P_m$ of $m$ vertices, the diameter increases to $m-1$. It follows that

$$D^{+0}(C_m) = 0 \quad \text{and} \quad D^{+k}(C_m) = 1 \text{ for } 1 \leqslant k \leqslant m-1-\lfloor m/2 \rfloor.$$

To find $D^{-k}(G)$, it suffices to consider adding edges to reduce the distance of all of farthest neighbors. Given a vertex $v$ in graph $G$, vertex $u$ is called a *farthest neighbor* of $v$, denoted as $v^f$, if $d_G(u,v) = D(G)$. Given a vertex $i$ in $C_m$ with $0 \leqslant i \leqslant \lfloor m/2 \rfloor$, farthest neighbor of $i$ is $i + \lfloor m/2 \rfloor$ if $m$ is even; $i + \lceil m/2 \rceil$ or $i + \lfloor m/2 \rfloor$ if $m$ is odd.

**Lemma 1.** $D^{-}(C_m) \geqslant 2$.

**Proof.** Suppose $D^{-}(C_m) = 1$. We can assume without loss of generality that adding an edge $e = (0,l)$ with $2 \leqslant l \leqslant \lfloor m/2 \rfloor$ reduces the diameter. Let $v = \lfloor l/2 \rfloor$ and $\bar{v} = v + \lfloor m/2 \rfloor$. Since the distance from $v$ to $\bar{v}$ is reduced, the edge $e$ must be used in the shortest path from $v$ to $\bar{v}$. It follows that

$$\begin{aligned} d_{C_m}(v,\bar{v}) &= \min\{d_{C_m}(v,0) + 1 + d_{C_m}(l,\bar{v}), d_{C_m}(v,l) + 1 + d_{C_m}(0,\bar{v})\} \\ &= \min\{\lfloor l/2 \rfloor + 1 + (\lfloor l/2 \rfloor + \lfloor m/2 \rfloor - l), (l - \lfloor l/2 \rfloor) + 1 + (m - \lfloor l/2 \rfloor - \lfloor m/2 \rfloor)\} \geqslant \lfloor m/2 \rfloor, \end{aligned}$$

which is a contradiction. Therefore $D^{-}(C_m) \geqslant 2$. $\square$

Now, we consider the case of adding two edges to cycle $C_m$. Let $D^{*}(C_m)$ denote the minimum diameter among those graphs obtained by adding two edges to $C_m$.

**Lemma 2.** *Let $m \geqslant 5$. Then,* $D^{*}(C_m) = \begin{cases} \lfloor m/4 \rfloor + 1 & \text{if } m \equiv 0,1,2 \textbf{ mod } 4, \\ \lfloor m/4 \rfloor + 2 & \text{if } m \equiv 3 \textbf{ mod } 4. \end{cases}$

**Proof.** Assume that we are adding two intersecting edges $e_1 = (0, l_1 + l_2)$ and $e_2 = (l_1, l_1 + l_2 + l_3)$ to the cycle. Let $G$ denote the resulting graph. The four endpoints of these two edges divide the cycle into four paths $Q_1, Q_2, Q_3$ and $Q_4$ of length $l_1, l_2, l_3$ and $l_4$, respectively, where $\sum_{i=1}^{4} l_i = m$. The four paths $Q_1, Q_2, Q_3$ and $Q_4$ are depicted in Fig. 1.