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Advances in type-2 fuzzy sets and systems

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Abstract

In this state-of-the-art paper, important advances that have been made during the past five years for both general and interval type-2 fuzzy sets and systems are described. Interest in type-2 subjects is worldwide and touches on a broad range of applications and many interesting theoretical topics. The main focus of this paper is on the theoretical topics, with descriptions of what they are, what has been accomplished, and what remains to be done.

Keywords: Type-2 fuzzy sets; Type-2 fuzzy systems; Interval type-2 fuzzy sets; Interval type-2 fuzzy logic systems; Centroid; KM algorithms; Computing with words; Fuzzy weighted average

1. Introduction

Type-2 fuzzy sets (T2 FS), which were introduced by Zadeh in [98], are now very well established and (as shall be demonstrated in this paper) are gaining more and more in popularity. In [51] we find answers to the following:

1. *Why did it take so long for the concept of a T2 FS to emerge?* It seems that science moves in progressive ways where one theory is eventually replaced or supplemented by another, and then another. In school we learn about determinism before randomness. Learning about type-1 (T1) FSs before T2 FSs fits a similar learning model. So, from this point of view it was very natural for fuzzyites to develop T1 FSs as far as possible. Only by doing so was it really possible later to see the shortcomings of such FSs when one tries to use them to model words or to apply them to situations where uncertainties abound.
2. *Why didn't T2 FSs immediately become popular?* Although Zadeh introduced T2 FSs in 1975, very little was published about them until the mid-to late nineties. Until then they were studied by only a relatively small number of people, including: [13,14,19,20,63,64,66,86]. Recall that in the 1970s people were first learning

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what to do with T1 FSs, e.g. fuzzy logic control. Bypassing those experiences would have been unnatural. Once it was clear what could be done with T1 FSs, it was only natural for people to then look at more challenging problems. This is where we are today.

3. *Why do we believe that by using T2 FSs we will outperform the use of T1 FSs?* T2 FSs are described by membership functions (MFs) that are characterized by more parameters than are MFs for T1 FSs. Hence, T2 FSs provide us with more design degrees of freedom; so using T2 FSs has the potential to outperform using T1 FSs, especially when we are in uncertain environments. Note that, at present, *there is no theory that guarantees that a T2 FS will always do this*.

One sign of a vibrant field is its applications. Here we categorize the applications that have appeared in the literature for T2 fuzzy sets and systems since 2001. For applications prior to that year, see [47, pp. 13–14].

> *Approximation:* [61] (TSK/steel strip temperature); *Clustering:* [26] (*C* spherical shells algorithm), [72] (fuzzy C-means); *Control:* [40] (marine and traction diesel engines), [75] (integrated development platform), [4] (evolutionary computing/NL dynamic plants), [36] (buck DC–DC converters), [17] (tracking mobile objects/robotic soccer games), [93] (liquid-level), [81] (proportional control), [25] (autonomous mobile robots), [24] (autonomous mobile robots/hierarchical), [44] (adaptive control of nonlinear plants); *Databases:* [67] (summarization); *Decision making:* [70] (variation in human decision making); *Embedded agents:* [10,11] (ambient intelligent environments); *Health care:* [27] (clinical diagnosis), [9] (differential diagnosis), [92] (nursing assessment); *Hidden Markov models:* [103] (phoneme recognition); *Neural networks:* [73] (fuzzy perceptron); *Noise cancellation:* [3] (adaptive noise cancellation); *Pattern classification:* [74] (fuzzy *k*-nearest neighbor); *Quality Control:* [45] (sound speakers); *Spatial query:* [71] (spatial objects); *Wireless communications:* [35] (wireless sensors/power on-off control), [77] (wireless sensor network lifetime analysis).

This paper focuses on advances in T2 fuzzy sets and systems since the year 2001, because earlier works are already well documented, e.g. [47]. The focus is on theoretical and computational issues. While some issues have been resolved, many new ones have been exposed, so T2 is a very fertile field for research.

Up until 2001, there was a very heavy emphasis on interval T2 FSs (IT2 FSs) and FLSs (IT2 FLSs), primarily because of their computational tractability. This emphasis has continued; however, interests have also turned towards more general kinds of T2 FSs and systems. Both T2 paths are covered in this paper. Section 2 covers topics about general T2 FSs and FLSs, and Section 3 covers topics about IT2 FSs and IT2 FLSs. Section 4 covers the fuzzy weighted average; Section 5 covers computing with words; and, Section 6 provides our conclusions.

It is assumed that the reader has some familiarity with T2 fuzzy sets and systems. For a relatively simple introduction to the former, see [54], and for the latter, see [47,55].

2. General T2 FSs and FLSs

In Section 2.1 we begin by presenting a Representation Theorem for a T2 FS. It is one of the most useful results in T2 FS theory because it can be used to derive many things that are associated with that theory, both old and new, in a simple and straightforward manner. Unfortunately, is not useful for computation; hence, the latter needs to be approached from other viewpoints. As for T1 FSs, the fundamental computations for T2 FSs are union, intersection and complement, and how to compute them, as well as attendant difficulties in such computations, are discussed in Section 2.2. One of the major applications for T2 FSs is a rule-based FLS, namely a T2 FLS, which is overviewed in Section 2.3. The major new calculation in a T2 FLS is called *type-reduction*; it maps a T2 FS into a T1 Fs, after which it is a simple matter to defuzzify the T1 FS in order to obtain a number at the output of the T2 FLS. Type-reduction, which is a major bottleneck for a T2 FLS, is overviewed in Section 2.4, and new ways for computing it are mentioned.

Zadeh [99–101] has introduced the *computing with words* (CWW) paradigm. Because words mean different things to different people, Mendel [50,51] has argued that words must be modeled using T2 FSs when computers interact with people and the interactions use FSs. In order to map from T2 FS word models back into a word, one will need the concept of similarity of T2 FSs, which is discussed in Section 2.5.