International Journal of Approximate Reasoning 76 (2016) 1–17

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Ordinal sum construction for uninorms and generalized uninorms

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

ARTICLE INFO

Article history:
Received 24 September 2014
Received in revised form 20 April 2016
Accepted 25 April 2016
Available online 28 April 2016

Keywords:
Ordinal sum
Uninorm
T-norm
T-subnorm
Additive generator
Quasi-inverse

ABSTRACT

The ordinal sum construction yielding uninorms is studied. A special case when all summands in the ordinal sum are isomorphic to uninorms is discussed and the most general semigroups that yield a uninorm via the ordinal sum construction, called generalized uninorms, are studied.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

The (left-continuous) t-norms and their dual t-conorms play an indispensable role in many domains such as probabilistic metric spaces [21], fuzzy logic [4], fuzzy control [22], non-additive measures and integrals [19], multi-criteria-decision making [26] and others. Each continuous t-norm is an ordinal sum of continuous Archimedean t-norms, and each continuous Archimedean t-norm possesses a continuous additive generator. However, in [9] (see also [10]) it was shown that the most general operations that yield a t-norm via the ordinal sum construction are t-subnorms.

In order to model bipolar behavior, uninorms were introduced in [24] (see also [3]). A uninorm U restricted to $[0, e]^2$, where e is the neutral element of U is a t-norm on $[0, e]^2$, and U restricted to $[e, 1]^2$ is a t-conorm on $[e, 1]^2$. Each uninorm is isomorphic to a bipolar t-conorm on $[-1, 1]$ (see [15]), i.e., a bipolar operation that is disjunctive with respect to the neutral point 0 (i.e., aggregated values diverge from the neutral point).

T-norms, t-conorms as well as uninorms are Abelian semigroups and therefore it is possible to apply the ordinal sum of Clifford for their construction. As uninorms are closely related to t-norms and t-conorms, it is clear that an ordinal sum that yields a uninorm will be closely connected with the ordinal sum that yields the corresponding underlying t-norm and t-conorm. In the case of t-norms (t-conorms) the basic stones in the ordinal sum construction are t-subnorms (t-superconorms). In this paper we investigate which operations can be used in the construction of uninorms via the ordinal sum and we call them generalized uninorms.

As we mentioned above, each continuous Archimedean t-norm possesses a continuous additive generator which has a range from $[0, \infty]$. Moreover, also t-subnorms can be additively generated. In the case of t-subnorms the strict monotonicity

E-mail address: zemankova@mat.savba.sk.

http://dx.doi.org/10.1016/j.ijar.2016.04.007
0888-613X/© 2016 Elsevier Inc. All rights reserved.

of additive generators of t-norms can be relaxed. If we consider a strictly monotone, continuous additive generator with the range $[-\infty, \infty]$ the generated operation will be a uninorm. Therefore our other interest is whether by relaxing the strict monotonicity of the additive generator of a uninorm we can generate a generalized uninorm.

The paper is structured as follows. In Section 2, some basic notions and results are recalled. The ordinal sum construction of Clifford is used to construct uninorms (Section 3) and a special case when all summands in this ordinal sum are isomorphic to uninorms is discussed in Section 4. In Section 5 we show the basic facts on generalized uninorms and in Section 6 we then study generated generalized uninorms. We give our conclusions in Section 7.

2. Basic notions and results

We will start with several important definitions (see [8,14]).

Definition 1.

(i) A triangular norm is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element.
(ii) A triangular conorm is a binary function $C\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.
(iii) A triangular subnorm is a binary function $M\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and there is $M(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$.
(iv) A triangular superconorm is a binary function $R\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and there is $R(x,y) \ge \max(x,y)$ for all $(x,y) \in [0,1]^2$.

Due to the associativity n-ary form of any t-norm (t-conorm) is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$.

The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm C by the equation

$$C(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa. The same duality holds between t-subnorms and t-superconorms.

Now let us recall an ordinal sum construction for t-norms and t-conorms [8].

Proposition 1. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ $((]c_k, d_k[)_{k\in K})$ be a system of open disjoint subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ $((C_k)_{k\in K})$ be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ $(C = (\langle c_k, d_k, C_k\rangle \mid k\in K))$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in [a_k, b_k[^2\,, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$C(x,y) = \begin{cases} c_k + (d_k - c_k)C_k\left(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}\right) & \text{if } (x,y) \in]c_k, d_k]^2\,, \\ \max(x,y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm C) is continuous if and only if all summands T_k (C_k) for $k \in K$ are continuous.

Proposition 2 *([8]). Let $t\colon [0,1] \longrightarrow [0,\infty]$ $(c\colon [0,1] \longrightarrow [0,\infty])$ be a continuous strictly decreasing (increasing) function such that $t(1) = 0$ $(c(0) = 0)$. Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ $(C\colon [0,1]^2 \longrightarrow [0,1])$ given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$
$$C(x,y) = c^{-1}(\min(c(1), c(x) + c(y)))$$

is a continuous t-norm (t-conorm). The function t (c) is called an additive generator *of T (C).*

An additive generator of a continuous t-norm T (t-conorm C) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm C) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e.,

Download English Version:

https://daneshyari.com/en/article/396877

Download Persian Version:

https://daneshyari.com/article/396877

Daneshyari.com