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Mixed aleatory and epistemic uncertainty quantification using fuzzy set theory

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ABSTRACT

This paper proposes algorithms to construct fuzzy probabilities to represent or model the mixed aleatory and epistemic uncertainty in a limited-size ensemble. Specifically, we discuss the possible requirements for the fuzzy probabilities in order to model the mixed types of uncertainty, and propose algorithms to construct fuzzy probabilities for both independent and dependent datasets. The effectiveness of the proposed algorithms is demonstrated using one-dimensional and high-dimensional examples. After that, we apply the proposed uncertainty representation technique to isocontour extraction, and demonstrate its applicability using examples with both structured and unstructured meshes.

1. Introduction

Numerical modeling and simulations help to study and predict the physical events or the behavior of engineered systems [1], and also reduce the need for costly or cumbersome physical experiments [2]. However, due to the inherent variability of the physical systems, the assumptions embodied in the mathematical models, nonspecified physical characteristics, the numerical approximations, etc., uncertainty is inevitable in the modeling and simulation process [3]. Therefore, to provide useful and reliable information regarding the physical system, understanding and quantifying the uncertainty in the simulations become critical.

Uncertainty in simulations, according to its various sources, can be broadly categorized into two types: *aleatory* and *epistemic*. Aleatory uncertainty (stochastic or irreducible uncertainty) describes the variability in the physical system, and variables with aleatory uncertainty can be treated as random variables. Epistemic uncertainty, known as systematic or reducible uncertainty, is considered to be a consequence of the incomplete or inadequate knowledge of the physical system [4].

Probability theory (or the concept of probability theory) is well developed and has been considered as the most suitable choice for aleatory uncertainty quantification. While alternative mathematical frameworks, such as fuzzy set theory [5,6], possibility theory [7,8], evidence theory [9,10], fuzzy measures as a unifying structure [11], and generalized p-boxes [12], are explored for possible better representations of epistemic uncertainty. Recently, it is becoming more common that prob-

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ability theory and other modern mathematical frameworks are combined together for uncertainty quantification due to the simultaneously presence of both aleatory and epistemic uncertainties in systems [3,13–17]. For example, Roy et al. implemented the mixed types of uncertainty propagation using probability theory coupled with interval analysis, and built a p-box to represent the propagated uncertainty in the model output [3]. Tang et al. coupled Dempster–Shafer theory and probability theory to quantify the mixed aleatory and epistemic uncertainty using a polynomial surrogate, and introduced local sensitivity analysis based on Dempster–Shafer theory [13]. Huang et al. dealt with the mixed types of uncertainty using both random variables and fuzzy variables in reliability based design optimization [16,18]. However, all the mentioned studies assume the mathematical representation for the uncertain input parameters, representing the mixed types of uncertainty in a given piece of information using different mathematical frameworks (i.e., modeling knowledge about uncertain input variables) is still under development.

In the current work, we focus on developing approaches for representing mixed types of uncertainty. Specifically, we consider the following scenario: Y is a random variable with unknown PDF and its associated piece of information is in the format of a finite size of an ensemble (i.e., a finite number of samples). In such a situation, both aleatory and epistemic uncertainties exist. To represent the mixed types of uncertainty mathematically, we adopt the relatively well-developed nonprobabilistic theory – fuzzy set theory – to construct fuzzy probabilities.

The concept of fuzzy probabilities (defining a possibility measure/fuzzy set over a probability value) has been investigated in the literature [19–21]. For example, Zadeh was the first to propose "fuzzy probabilities" where probabilities are assumed to be fuzzy rather than real numbers [19]. They were defined in the situation where the propositions are represented by fuzzy sets. Buckley [20] has discussed the construction of "fuzzy probabilities" using nested confidence intervals and analyzed their properties from a theoretical point of view. Baudrit and Dubois [21] also mentioned the construction of possibility distribution based on nested confidence intervals for a presentation of incomplete probabilistic knowledge. In the current work, we propose simple methods for constructing fuzzy probabilities specifically aimed at mixed types of uncertainty representation and discuss the requirements of fuzzy probabilities in the field of uncertainty quantification. The main contributions of this study can be summarized as:

- Propose an uncertainty representation technique for modeling mixed types of uncertainty using the concept of fuzzy probabilities.
- Discuss the requirements of fuzzy probabilities for uncertainty representation in an ensemble.
- Demonstrate the properties of proposed uncertainty representation technique in both one-dimensional and high-dimensional examples.
- Apply the proposed uncertainty representation technique to isocontour extraction.

The remainder of the paper proceeds as follows. In Section 2, we provide a brief introduction to fuzzy set theory, where we introduce the membership function, the extension principle. Section 3 is devoted to the introduction of an uncertainty representation technique based on fuzzy probability, where we state the problem, and propose the requirements and algorithms for constructing fuzzy probabilities to model mixed types of uncertainty. We then demonstrate the properties of the proposed uncertainty representation technique using one-dimensional and high-dimensional examples in Section 4. In Section 5, we apply the proposed uncertainty representation technique to isocontour extraction; specifically, we extend the marching cubes algorithm to the fuzzy probabilistic marching cubes algorithm and demonstrate its effectiveness on a few examples. We summarize and conclude our work in Section 6.

2. Basics of fuzzy set theory

Fuzzy sets can be considered as a generalization of classical sets. The change of membership of each element in a fuzzy set is gradual and represented using a membership function.

Definition 1. Let U be a classical nonempty set and $x \in U$ be an element. A fuzzy set $\tilde{A} \subset U$ is defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in U\}, \quad (1)$$

where $\mu_{\tilde{A}}(x) : U \to [0, 1]$ is called the membership function of $\tilde{A}$.

The membership function $\mu_{\tilde{A}}(x)$ ($\mu_{\tilde{A}}$ for simplicity) represents the degree of the membership of the element x in the fuzzy set $\tilde{A}$. The element x is considered as a full member of the fuzzy set $\tilde{A}$ if $\mu_{\tilde{A}}(x) = 1$; whereas x is considered as not a member of the fuzzy set $\tilde{A}$ if $\mu_{\tilde{A}}(x) = 0$. A fuzzy set $\tilde{A}$ is called normal when $\sup_{x \in U} \mu_{\tilde{A}}(x) = 1$.

A fuzzy set can be characterized using classical sets, such as support and α-cuts.

$$\text{Support:} \quad \mathrm{Supp}(\tilde{A}) = \{x | \mu_{\tilde{A}}(x) > 0\}, \quad (2)$$

$$\alpha\text{-cut:} \quad [\tilde{A}]^{\alpha} = \{x | \mu_{\tilde{A}}(x) \geq \alpha\}. \quad (3)$$