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Central points and approximation in residuated lattices

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ABSTRACT

The paper presents results on approximation in residuated lattices given that closeness is assessed by means of biresiduum. We describe central points and optimal central points of subsets of residuated lattices and examine several of their properties. In addition, we present algorithms for two problems regarding optimal approximation.

1. Introduction and preliminaries

Functions on lattices related to concepts of magnitude and distance, such as valuations and metrics, received considerable attention in lattice theory, see e.g. [4, Chap. II and X]. In this paper, we study certain problems related to closeness in complete residuated lattices, which is represented by the so-called biresiduum. Recall that a complete residuated lattice [16] is a structure $\mathbf{L} = \langle L, \otimes, \rightarrow, \wedge, \vee, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ (multiplication) and $\rightarrow$ (residuum) form an adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$. Examples of residuated lattices are abundant in mathematics and logic [8,13]. In what follows, we use the following three well-known examples of complete residuated lattices on $L = [0, 1]$ induced by continuous t-norms [14]: the standard Łukasiewicz algebra ($a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$), the standard product algebra ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = b/a$ otherwise), also called the standard Goguen algebra, the standard Gödel algebra ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, and $a \rightarrow b = b$ otherwise). It is well known [1,12] that a biresiduum $\leftrightarrow$, defined in any residuated lattice by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a),$$

satisfies

$$a \leftrightarrow b = 1 \quad \text{iff} \quad a = b, \tag{1}$$

$$a \leftrightarrow b = b \leftrightarrow a, \tag{2}$$

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq a \leftrightarrow c. \tag{3}$$

Hence, $a \leftrightarrow b$ may naturally be interpreted as an element in L representing a degree of closeness (similarity) of a and b, with $(a_1 \leftrightarrow b_1) \leq (a_2 \leftrightarrow b_2)$ meaning that a_2 and b_2 are closer (more similar) to each other than a_1 and b_1. Note that (1)–(3) resemble dual versions of the axioms of a metric with a generalized triangular inequality. Indeed, for the standard

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Łukasiewicz algebra, $d_{\leftrightarrow}(a,b) = 1 - (a \leftrightarrow b)$ is a $[0,1]$-valued metric on $[0,1]$. This metric coincides with the usual Euclidean metric and is called the Chang distance function [6]. More generally, using $d_{\leftrightarrow}(a,b) = f(a \leftrightarrow b)$ one obtains a metric from a biresiduum of a continuous Archimedean t-norm with an additive generator f [14]. For the standard Gödel algebra, in which case the t-norm is not Archimedean, $d_{\leftrightarrow}(a,b) = 1 - (a \leftrightarrow b)$ is a $[0,1]$-valued ultrametric on $[0,1]$. General relationships between fuzzy equivalences and metric structures along this line, including metrics, ultrametrics, and in general so-called G-metrics, are found in [15]. These relationships show that similarities represented by biresidua are closely related to metric-like structures and are richer than the ordinary metrics.

Note also that residuated lattices and their generalizations, developed initially within the studies of ring ideals [16], are the main structures of truth degrees used in many-valued logic [9–11] in which biresiduum is the truth function of the logical connective of equivalence.

In this paper, we present results motivated by the following problem. Given a set of elements of a residuated lattice, what are its central points, i.e. elements which are close/similar (as much as possible or to some specified level) to every element of the set, provided closeness/similarity is assessed by means of biresiduum?

2. Preliminaries

We assume familiarity with some basic properties of residuated lattices [16] and basic concepts from tolerance relations on complete lattices [7,17]. In this section, we recall briefly what we use in the paper.

Each residuated lattice satisfies the following conditions:

$$(a \otimes b) \to c = a \to (b \to c), \tag{4}$$

$$a \otimes (a \to b) \leq b, \tag{5}$$

$$a \to (a \otimes b) \geq b. \tag{6}$$

Moreover,

$$a_1 \leq a_2 \text{ and } b_1 \leq b_2 \quad \text{implies} \quad a_1 \otimes b_1 \leq a_2 \otimes b_2 \tag{7}$$

$$\text{and} \quad a_2 \to b_1 \leq a_1 \to b_2. \tag{8}$$

The following conditions are satisfied in each complete residuated lattice:

$$a \to (\textstyle\bigwedge_{b\in B} b) = \textstyle\bigwedge_{b\in B}(a \to b), \tag{9}$$

$$(\textstyle\bigvee_{b\in B} b) \to a = \textstyle\bigwedge_{b\in B}(b \to a), \tag{10}$$

$$a \to (\textstyle\bigvee_{b\in B} b) \geq \textstyle\bigvee_{b\in B}(a \to b), \tag{11}$$

$$(\textstyle\bigwedge_{b\in B} b) \to a \geq \textstyle\bigvee_{b\in B}(b \to a), \tag{12}$$

$$a \otimes (\textstyle\bigvee_{b\in B} b) = \textstyle\bigvee_{b\in B}(a \otimes b), \tag{13}$$

$$a \otimes (\textstyle\bigwedge_{b\in B} b) \leq \textstyle\bigwedge_{b\in B}(a \otimes b). \tag{14}$$

A *tolerance* is a reflexive and symmetric binary relation. A *block* of a tolerance T on a set U is a subset B of U for which $B \times B \subseteq T$. A maximal block of T is a block B of T which is maximal with respect to set inclusion. A collection of maximal tolerance blocks of T is denoted by U/T and forms a covering of U. A *class* of T given by $u \in U$ is the set $[u]_T = \{v \in U \mid u\,T\,v\}$. Clearly, if T is an equivalence, maximal blocks as well as classes of T are just the equivalence classes of T.

Throughout the paper, $\mathbf{L}$ denotes a complete residuated lattice and e an element of its support set L. By $\approx_e$, we denote the tolerance on L defined by

$$a \approx_e b \quad \text{iff} \quad a \leftrightarrow b \geq e.$$

3. Central points, central sets, and maximal blocks

For $B \subseteq L$, we set

$$C_e(B) = \{c \in L \mid \text{for each } b \in B, c \leftrightarrow b \geq e\}. \tag{15}$$

We call $C_e(B)$ the *e-central set of B* and its elements *e-central points of B*.

Lemma 3.1. *$c \in C_e(B)$ iff $(c \to \bigwedge B) \wedge (\bigvee B \to c) \geq e$.*

Proof. (9) and (10) yield $c \to (\bigwedge_{b\in B} b) = \bigwedge_{b\in B}(c \to b)$ and $(\bigvee_{b\in B} b) \to c = \bigwedge_{b\in B}(b \to c)$. $\square$

Denoting by $[p,q]$ the interval $\{x \in L \mid p \leq x \leq q\} \subseteq L$, we get: