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Probabilism, entropies and strictly proper scoring rules

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ABSTRACT

Accuracy arguments are the en vogue route in epistemic justifications of probabilism and further norms governing rational belief. These arguments often depend on the fact that the employed inaccuracy measure is strictly proper. I argue controversially that it is ill-advised to assume that the employed inaccuracy measures are strictly proper and that strictly proper statistical scoring rules are a more natural class of measures of inaccuracy. Building on work in belief elicitation I show how strictly proper statistical scoring rules can be used to give an epistemic justification of probabilism.

An agent's evidence does not play any role in these justifications of probabilism. Principles demanding the maximisation of a generalised entropy depend on the agent's evidence. In the second part of the paper I show how to simultaneously justify probabilism and such a principle. I also investigate scoring rules which have traditionally been linked with entropies.

Introduction and Notation

1. Introduction

All Bayesians agree on one basic norm governing strength of rational belief

Probabilism. Any rational agent's subjective belief function ought to satisfy the axioms of probability and every probability function is, in principle, permissible. *Prob*

The question arises as to how to justify this norm. Traditionally, axiomatic justifications [6,41], justifications on logical grounds [22] and Dutch Book Arguments [12,50] were given to this end. Dutch Book Arguments have been widely regarded as the most persuasive justification, however, they have recently begun losing some of their once widespread appeal [21].[1]

Recent epistemic justifications of probabilism are accuracy-based arguments [24,25,30,31,49], which all build on [11]. The latter three arguments employ Inaccuracy Measures (IMs) which are assumed to be strictly proper. These IMs are closely related to the notion of a *Scoring Rule* (SR) which the statistical community has a long tradition of studying, see [10] in the Encyclopedia of Statistics.

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[1] We are joining the debate concerning rational belief formation assuming that degrees of beliefs are best represented by real numbers in the unit interval $[0,1] \subset \mathbb{R}$. Anyone who rejects this premise will have to carefully assess whether the here presented account has implications on her line of thinking. Some of our results also hold true for degrees of belief represented by arbitrary positive real numbers.

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In the first part of this paper, we argue that statistical SRs, properly understood, are better suited than IMs to justify *Prob*. The argument will be along the following lines: the most convincing justifications of *Prob* relying on IMs require these IMs to be strictly proper (Section 4.1). However, for the purposes of justifying *Prob*, assuming that an IM is strictly proper is ill-advised (Section 4.3). On the contrary, assuming that an SR is strictly proper is not only defensible but a desideratum (Section 3.2).

In Theorem 5.6 we show how strictly proper IMs give rise to strictly proper SRs in a canonical way. We demonstrate in Theorem 6.2 how the class of so-constructed SRs can be used to justify *Prob*.

The justifications in the first part of this paper do not take the agent's evidence into account. In all realistic cases rational agents do possess some evidence and this evidence ought to influence their degrees of belief, in some way. Maximum (generalised) entropy principles require an agent to adopt the probability function which maximises (a generalised) entropy among those probability functions which satisfy constraints imposed by her evidence.

In the second part of this paper we show how to simultaneously justify *Prob* and a such principle (Theorem 7.1 and Theorem 7.2). The usual argument here consists of a two-stage justification – first one justifies *Prob* and then one justifies the entropy principle – and a story explaining why and how the justification of *Prob* trumps that of the entropy principle. The advantage of the simultaneous justification given here is that no such story needs to be told.

Taken together, *Prob* and such a principle entail the Principle of Indifference (PoI) in a large number of cases (Theorem 7.5, Corollary 7.6).

The logarithmic SR is well-known to be the only local SR which is strictly proper when applied to belief functions which are probability functions. Furthermore, this SR is at the heart of the maximum entropy principle. Since we here do not presuppose *Prob*, we investigate notions of locality applied to SRs for general belief functions (Section 8 and Section 9). We prove a non-existence result for such SRs in Theorem 8.4. Furthermore, we investigate how to weaken our assumptions to obtain strictly proper statistical SRs which are local in some sense, see Proposition 9.1 and Proposition 9.2.

2. The formal framework

Throughout, we work with a fixed, non-empty and finite set Ω, which is interpreted as the set possible worlds or elementary events. The power set of Ω, $\mathcal{P}\Omega$, is the set of events or the set of propositions. We shall assume throughout that $|\Omega| \geq 2$ and for $X \subseteq \Omega$ let $\bar{X} := \Omega \setminus X$.

The set of probability functions $\mathbb{P}$ is the set of functions $P : \mathcal{P}\Omega \to [0, 1]$ such that $\sum_{\omega\in\Omega} P(\{\omega\}) = 1$ and whenever $X \subseteq \Omega$ is such that $X = Y \cup Z$ with $Y \cap Z = \emptyset$, then $P(X) = P(Y) + P(Z)$. We shall use $P(\omega)$ as shorthand for $P(\{\omega\})$.

Note that for all probability functions $P \in \mathbb{P}$ we have that $P(X) + P(\bar{X}) = 1$ and hence $2\sum_{X\subseteq\Omega} P(X) = \sum_{X\subseteq\Omega} P(X) + P(\bar{X}) = |\mathcal{P}\Omega|$.

The set of belief functions is the set of functions $Bel : \mathcal{P}\Omega \to [0, 1]$ and shall be denoted by $\mathbb{B}$. Throughout, we assume that all belief and probability functions are *total*, i.e. defined on every $X \subseteq \Omega$. Trivially, since $|\Omega| \geq 2$ we have $\mathbb{P} \subset \mathbb{B}$, where $\subset$ denotes strict inclusion. Of particular interest are the functions $v_\omega \in \mathbb{P}$ for $\omega \in \Omega$. A v_ω is the *at a world* $\omega \in \Omega$ *vindicated credence function*. A v_ω can also be thought of as the *indicator function of the elementary event* $\omega \in \Omega$. The v_ω are defined as follows:

$$v_\omega(X) := \begin{cases} 0 & \text{if } X \text{ is false at } \omega \\ 1 & \text{if } X \text{ is true at } \omega. \end{cases}$$

By "X is true at ω" we mean that $\omega \in X$; on the contrary, "X is false at ω", if and only if $\omega \notin X$.

In this paper we will stay within the classical framework of decision making developed in [53]. So, we assume act-state independence,[2] we also only consider propositions which do not refer to themselves nor to their chances. Such propositions are well-known to cause problems for the classical decision making framework. Unsurprisingly, accuracy arguments based on the classical decision making framework are also troubled by such propositions, see [5,18]. Decision making frameworks for accuracy arguments which can deal with such propositions are explored in [27].

Part 1.

3. The statistical approach

3.1. Scoring rules, applications and interpretations

Central to SRs and IMs is a measure function measuring the goodness or badness, in some sense, of a belief function *Bel*. In the statistical community this function is interpreted pragmatically as a loss incurred in a betting scenario, whereas the epistemic tradition interprets the goodness measure as a measure of (in)accuracy.

SRs have mainly been used to *elicit beliefs* or to *assess forecasts*. For belief elicitation it is widely assumed that the agent's belief function Bel^* is a probability function, i.e., $Bel^* \in \mathbb{P}$. Similarly, forecasted events are normally assumed to be ruled

[2] In our context this means that neither the truth value nor the objective probability of a proposition $X \subseteq \Omega$ depends on the agent's belief function *Bel*.