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Accept & reject statement-based uncertainty models

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ABSTRACT

We develop a framework for modelling and reasoning with uncertainty based on accept and reject statements about gambles. It generalises the frameworks found in the literature based on statements of acceptability, desirability, or favourability and clarifies their relative position. Next to the statement-based formulation, we also provide a translation in terms of preference relations, discuss—as a bridge to existing frameworks—a number of simplified variants, and show the relationship with prevision-based uncertainty models. We furthermore provide an application to modelling symmetry judgements.

1. Introduction

1.1. *What & why*

Probability theory and the statistical tools built on it help us deal with uncertainty, be it caused by the lack of information about or by the variability of some phenomenon. Probability theory provides a mathematical framework for modelling uncertainty that is centred around the quantification of the uncertainty. It also provides rules for reasoning under uncertainty, i.e., deductive inference: how to go from an assessment, such as probability values for some events, to conclusions and decisions, e.g., the expected value of some quantity or the selection of an optimal action.

In this paper, we present a mathematical framework for modelling uncertainty that generalises probability theory and that is centred around the categorisation of gambles—i.e., quantities about whose value we are uncertain—into acceptable and rejected ones. Crudely speaking, gambles whose expectation is assessed to be non-negative are acceptable and those whose expectation is assessed to be negative are rejected. Our mathematical framework includes rules for reasoning under uncertainty, but now the assessments we start from consist of sets of acceptable and rejected gambles. We do not discuss how the assessments are obtained from domain experts or experimental data and restrict ourselves to the unconditional case.

Why did we develop this theory? The main goal is to unify many of the existing generalisations of probability theory that are explicitly or implicitly based on the assumption that gamble values can be expressed in a linear precise utility. Of these

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generalisations, some essentially express uncertainty using a *strict* (partial) order of a set of gambles of interest, which is useful for decision making applications, and some use a *non-strict* (partial) order, which is practical when expressing indifference judgements such as those arising when modelling symmetry assumptions. Because of the unifying character of our theory, we can combine the strengths of each type of theory and also express the relative position of the models representable within each of the theories. A consequence is of course that our theory is more expressive than the theories we unify.

We have already mentioned that the representation in our theory consists of a pair of gamble sets. This type of representation differs markedly and is far less common than the typical, functional one, in which one works with functions that map events and gambles to probability and expectation values. It is closely related to, but less popular than the preference order one, where, roughly speaking, events are ranked according to likelihood and gambles according to expectation. Our experience with working with representations using sets of gambles has convinced us that it deserves more attention: the formulation of theory and derivation of results is mostly simplified to intuitive applications of basic set theory, linear algebra, and convex analysis. A side goal of this paper is therefore to bring attention to this representation by putting it centre stage.

Although it is not the focus of this paper, it is useful to point out another advantage of the theory we develop: Because it generalises theories using different representations of uncertainty, the elicitation statements that go along with these representations—probabilities for events, preferences between gambles, etc.—can all be incorporated.

1.2. Literature context

We have drawn much inspiration from the foundational works of the theories of imprecise probability [1–9], which all generalise probability theory in a similar way. In these theories, one can specify lower and upper probabilities and expectations instead of just unique, precise probability and expectation values. The introduction of these new concepts is justified by giving them a clear interpretation and by the fact that often not enough information is available to fix unique, precise values. Our theory possesses the expressiveness to deal with imprecision, as these theories of imprecise probability do.

Probability measures, the models of classical probability theory, correspond to complete preference orders. This means that all events or gambles can be compared by their unique probability or expectation value. This is no longer the case in theories of imprecise probabilities, where events and gambles may be uncomparable and whose models may correspond to partial preference orders. Sometimes strict preference is taken as basic (see, e.g., [9–11]) and sometimes non-strict preference is (see, e.g., [9,12]); the other relation may then be derived from it (see, e.g., [13]). We can associate both a strict and a non-strict preference order with each of the models in our theory. But now neither is basic and each can essentially be specified separately, but both are related with each other in a natural way. This is not the only way to generalise things; Seidenfeld et al. [14,15] use strict preference, choose their axioms in view of characterising the preference order in terms of sets of functionals and move to preference between sets of gambles in their coherent choice function approach.

In this paper, we represent uncertainty using sets of gambles. Although this is as of yet uncommon in the literature, we are not the first. Smith [4, §14] uses them as a useful intermediate representation in the course of a proof in the exposition of his theory of imprecise probabilities. In the work of Williams [12,16], they become more prominent—he talks about sets of acceptable bets—, but he still keeps a focus on (non-linear) expectation-type models. Seidenfeld et al. [10, §IV] talk about 'favorable' gambles while presenting results about the relationship between convex sets of probability measures and strict preference orders. It is Walley [9,11]—talking about sets of desirable gambles—who seems to have been the first to discuss them in their own right. Compared to these authors, our contribution lies in considering a pair of sets instead of just one set, proposing an intuitive axiom of how both sets interact, and developing the resulting theory. Adding an extra set is what allows us to simultaneously represent strict and non-strict preferences.

In our framework, our first main axiom defines which assessments are irrational and which are not. The two other main axioms are generative in the sense that we can associate an extension operator with each of them that modifies assessments in such a way that the result satisfies the axiom. In spirit, this is completely analogous to the approach taken by de Finetti [17,18] in his development of probability theory and Walley [9] in his development of imprecise probability theory: they call their irrationality axiom 'avoiding sure loss', their (single) generative principle 'coherence', and the associated extension operator 'the fundamental theorem of probability' and 'natural extension', respectively.

The basic setup and the terminology we use in our paper shows the influence of the subjectivist school of de Finetti. We do not want this to be construed as a constraint. The results of this paper—its mathematical models and techniques—are applicable under different interpretations, analogously to the role measure theory takes in probability theory. In this context, the 'agent' that provides us with an assessment can, e.g., be seen as a person giving a 'subjective' opinion or an algorithm that transforms observations such as sample sequences into 'objective' opinions. We do not discuss how these opinions are elicited or constructed.

1.3. Overview

As stated, the prime goal of this paper is to present a mathematical framework for modelling uncertainty. The basic conceptual set-up and mathematical notation is given in Section 1.4. Our presentation starts from the basics and there is no prerequisite knowledge of (imprecise) probability theory, although of course this does not hurt. However, throughout, a basic familiarity with set theory, linear algebra, and convex analysis is assumed. Once the framework is established in Section 2,