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Interval MV-algebras and generalizations ☆

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ABSTRACT

For any MV-algebra A we equip the set $I(A)$ of intervals in A with pointwise Łukasiewicz negation $\neg x = \{\neg\alpha \mid \alpha \in x\}$, (truncated) Minkowski sum $x \oplus y = \{\alpha \oplus \beta \mid \alpha \in x, \ \beta \in y\}$, pointwise Łukasiewicz conjunction $x \odot y = \neg(\neg x \oplus \neg y)$, the operators $\Delta x = [\min x, \min x]$, $\nabla x = [\max x, \max x]$, and distinguished constants $0 = [0, 0], 1 = [1, 1], \mathrm{i} = A$. We list a few equations satisfied by the algebra $\mathcal{I}(A) = (I(A), 0, 1, \mathrm{i}, \neg, \Delta, \nabla, \oplus, \odot)$, call *IMV-algebra* every model of these equations, and show that, conversely, every IMV-algebra is isomorphic to the IMV-algebra $\mathcal{I}(B)$ of all intervals in some MV-algebra B. We show that IMV-algebras are categorically equivalent to MV-algebras, and give a representation of free IMV-algebras. We construct *Łukasiewicz interval logic*, with its coNP-complete consequence relation, which we prove to be complete for $\mathcal{I}([0, 1])$-valuations. For any class Q of partially ordered algebras with operations that are monotone or antimonotone in each variable, we consider the generalization $\mathcal{I}_{\mathsf{Q}}$ of the MV-algebraic functor $\mathcal{I}$, and give necessary and sufficient conditions for $\mathcal{I}_{\mathsf{Q}}$ to be a categorical equivalence. These conditions are satisfied, e.g., by all subquasivarieties of residuated lattices.

1. Introduction

In [37, §1.6] "truth-values" in (infinite-valued, propositional) Łukasiewicz logic $Ł_\infty$ are identified with values of measurements of *normalized bounded* physical observables, just as boolean truth-values essentially arise from $\{0, 1\}$-observables. Pursuing this line of interpretation, in this paper we will take into account the fact that the measurement of any continuous physical observable is affected by an error: that's why (up to normalization) measuring a bounded continuous observable amounts to specifying a closed interval in $[0, 1]$. A fortiori, if we regard our everyday estimations/evaluations as a generalization of the measurements of physical observables, then intervals provide more realistic truth-values than real numbers. Upon equipping these intervals with a judiciously chosen implication operation, and proceeding by analogy with what boolean logic does for $\{0, 1\}$ (no–yes) truth-values, we will construct a "Łukasiewicz interval logic" suitable for the formal handling of these generalized measurements.

Surely enough, Łukasiewicz interval logic has many forerunners, including full fledged logical systems — with their carefully crafted axioms, rules, consequence relations, and completeness theorems — whose truth-values are intervals rather than numbers (see, e.g., [14,19,26] and references therein; also see the pioneering paper [45] for motivation, and Section 9 for a more articulated discussion). While, on the one hand, the relative efficiency of operations on intervals accounts for

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the successful application of interval-valued truth-degrees in knowledge-based expert systems, [29], on the other hand, general systems for the formal handling of interval truth-values sometimes pose conceptual problems, ranging from the role of truth-functionality [43, Remark 33] to the significance of the truth-values themselves [18], and the mutual relations between implication and consequence [4,15]. Using Łukasiewicz logic $Ł_\infty$ as a template, in this paper we aim at developing the algebraic and categorical tools for the systematic construction of interval logics.

Following the MV-algebraic approach of [13], negation $1-x$ and truncated addition $x \oplus y = \min(1, x+y)$ are selected as the basic operations on truth-values in $Ł_\infty$, together with the derived conjunction operation $x \odot y = \max(0, x+y-1)$ and the distinguished constants 0 for "false" and 1 for "true". Łukasiewicz *implication* is derived by writing $x \to y = \neg x \oplus y$. Equivalently, implication can be taken as a basic operation, whence $\neg x = x \to 0$, $x \oplus y = \neg x \to y$, $1 = \neg 0$. Then the $\to$ operation is characterized among all binary operations on $[0,1]$ by the Smets–Magrez theorem, [1,41], as the only $[0,1]$-valued continuous map on $[0,1]^2$ satisfying natural monotonicity conditions with respect to the usual order of $[0,1]$. These conditions immediately yield the classical Łukasiewicz axioms of $Ł_\infty$. Via Modus Ponens, these axioms in turn determine the (coNP-complete) *$Ł_\infty$-consequence relation.* Closing a circle of ideas concerning truth-values, we recover the intended meaning of $Ł_\infty$-formulas by a classical *completeness theorem* [11,40], [13, §2.5] stating that $Ł_\infty$-tautologies (i.e., $Ł_\infty$-consequences of the Łukasiewicz axioms computed via Modus Ponens) coincide with formulas taking value 1 for every $[0,1]$-valuation.

Next, mimicking the approach to Łukasiewicz logic sketched in the foregoing paragraph, the set $I(A)$ of intervals in any MV-algebra A is equipped with pointwise Łukasiewicz negation $\neg x = \{\neg\alpha \mid \alpha \in x\}$, (truncated) *Minkowski sum*, $x \oplus y = \{\alpha \oplus \beta \mid \alpha \in x, \beta \in y\}$, pointwise Łukasiewicz conjunction $x \odot y = \neg(\neg x \oplus \neg y)$, the operators $\Delta x = [\min x, \min x]$, $\nabla x = [\max x, \max x]$, and distinguished constants $0 = [0,0]$, $1 = [1,1]$, $\mathrm{i} = [0,1] = A$. Differently from the operations defined in interval algebras and interval logics, our operations are like those traditionally considered in interval analysis, [31]. And yet, our paper pertains to the domain of approximate reasoning, rather than to interval analysis, because of the following constructions and results:

In (9)–(27) we list a few simple equations that are satisfied by the algebra $\mathcal{I}(A) = (I(A), 0, 1, \mathrm{i}, \neg, \Delta, \nabla, \oplus, \odot)$. We call *IMV-algebra* every model of these equations. The adequacy of these equations is shown in the *Representation Theorem (Theorem 3.5)* stating that, conversely, every IMV-algebra is isomorphic to the IMV-algebra $\mathcal{I}(B)$ of all intervals in some MV-algebra B. While for no IMV-algebra, its $(0, 1, \neg, \oplus, \odot)$-reduct is an MV-algebra, the functor $\mathcal{I}$ establishes a *categorical equivalence* between MV-algebras and IMV-algebras (Theorem 4.4).

We then prove the following equational *Completeness Theorem (Theorem 5.1)*: an equation is satisfied by all IMV-algebras iff it is satisfied by the IMV algebra $\mathcal{I}([0,1])$ of all intervals in the standard MV-algebra $[0,1]$ iff it is derivable from the IMV-axioms (9)–(27) by the familiar rules of replacing equals by equals according to Birkhoff equational logic. This result has a deeper algebraic counterpart in the representation (Theorem 5.2) of free IMV-algebras.

Every IMV-algebra J is naturally equipped with *two different partial orders*: the product order and the inclusion order. The former endows J with a distributive lattice structure $(J, \sqcup, \sqcap)$; the latter yields an upper semilattice structure $(J, \cup)$. In the final part of Section 5 is it proved that each operation $\sqcup, \sqcap, \cup$ is definable from the IMV-structure. With reference to our initial remarks on truth-values as intervals, IMV-algebras can express the fundamental inclusion order $u \subseteq v$ between actual measurements/estimations u, v, meaning that u is more precise than v. Within the MV-algebraic framework, the inclusion order has no meaning, despite MV-algebras are categorically equivalent to IMV-algebras.

In Section 7 we will introduce *Łukasiewicz interval logic*, with its consequence relation based on the only rule of Modus Ponens, and prove a completeness theorem for $\mathcal{I}([0,1])$-valuations, along with a (local) Deduction Theorem (Theorem 7.1). The consequence problem in this logic, just like the equational theory of IMV-algebras, turns out to be coNP-complete (Theorem 6.1, Corollary 7.2). Thus, the increased expressive power of Łukasiewicz interval logic with respect to Łukasiewicz logic does not entail greater complexity of the consequence problem.

In Section 8, for any class Q of partially ordered algebras with operations that are monotone or antimonotone in each variable, we consider the generalization $\mathcal{I}_{\mathsf{Q}}$ of the MV-algebraic functor $\mathcal{I}$, and give necessary and sufficient conditions for $\mathcal{I}_{\mathsf{Q}}$ to be a categorical equivalence. As shown in Corollary 8.12, these conditions are satisfied, e.g., by every quasivariety Q having a lattice reduct — including many classes of ordered algebras related with logical systems of general interest, such as BL-algebras, Heyting algebras, Gödel algebras, MTL-algebras, and more generally every subquasivariety of residuated lattices (Corollary 8.13).

Remarkably enough, the pervasiveness of the categorical equivalence $\mathcal{I}_{\mathsf{Q}}$ has gone virtually unnoticed in the literature on interval and triangle algebras and their logics, interval constructors and triangularizations. The final Section 9 is devoted to relating our results to the extensive literature on these topics, [2–4,14,15,42,43].

About Łukasiewicz interval logic and approximate reasoning. Both monographs [13] and [37] show that the rich algorithmic and mathematical structure of Łukasiewicz logic $Ł_\infty$ yields a useful tool for a mode of reasoning that differs from the classical mode, but still has a computable consequence relation satisfying a completeness theorem. Further, $Ł_\infty$ has a "first-order" extension, and allows a probability theory for many-valued events, via the notion of a "state". In view of the initial remarks in this section, an even more natural basis for approximate reasoning on the outcome of generalized measurements/estimations may be obtained from Łukasiewicz interval logic: this is a logical system where the value of a proposition is an interval, just as the output of a measurement of a physical observable; moreover, syntax matches semantics via a completeness theorem, implication matches consequence via a deduction theorem, and consequence has the same computational complexity as boolean consequence.